MAT 137Y, 2008-2009 Winter Session, Solutions to Term Test 1

1. Evaluate the following limits. (Do not prove them using the formal definition of limit.)

(10%) **(i)** $\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$.

Multiplying top and bottom by $1+\cos x$, we have

$$\begin{aligned}\lim_{x\to 0} \frac{x\sin x(1+\cos x)}{(1-\cos x)(1+\cos x)} &= \lim_{x\to 0} \frac{x\sin x(1+\cos x)}{1-\cos^2 x} = \lim_{x\to 0} \frac{x\sin x(1+\cos x)}{\sin^2 x} \\ &= \lim_{x\to 0} \frac{x}{\sin x}\cdot(1+\cos x) = 1\cdot 2 = 2.\end{aligned}$$

(10%) **(ii)** $\lim_{x\to 3} \frac{\sqrt{5-x}-\sqrt{x^2-7}}{\sqrt{x+6}-3}$.

Here we multiply top and bottom by the conjugates of both expressions to get

$$\begin{aligned}&\lim_{x\to 3} \frac{\sqrt{5-x}-\sqrt{x^2-7}}{\sqrt{x+6}-3}\left(\frac{\sqrt{5-x}+\sqrt{x^2-7}}{\sqrt{5-x}+\sqrt{x^2-7}}\right)\left(\frac{\sqrt{x+6}+3}{\sqrt{x+6}+3}\right) \\ &= \lim_{x\to 3} \frac{[(5-x)-(x^2-7)](\sqrt{x+6}+3)}{[(x+6)-9][\sqrt{5-x}+\sqrt{x^2-7}]} = \lim_{x\to 3} \frac{(12-x-x^2)(\sqrt{x+6}+3)}{(x-3)(\sqrt{5-x}+\sqrt{x^2-7})} \\ &= \lim_{x\to 3} \frac{(3-x)(4+x)(\sqrt{x+6}+3)}{(x-3)(\sqrt{5-x}+\sqrt{x^2-7})} = -\lim_{x\to 3} \frac{(4+x)(\sqrt{x+6}+3)}{(\sqrt{5-x}+\sqrt{x^2-7})} = -\frac{21}{\sqrt{2}}.\end{aligned}$$

2.

(10%) **(i)** Find all solutions in the interval $[0,2\pi)$ that satisfy the equation $2\sin 3x - 1 = 0$.

We have $2\sin 3x - 1 = 0 \iff \sin 3x = \frac{1}{2}$. Solving for $3x$, we have either $3x = \frac{\pi}{6} + 2k\pi$ or $3x = \frac{5\pi}{6} + 2k\pi$. Solving for x gives us the solutions

$$x = \frac{\pi}{18} + \frac{2}{3}k\pi, \qquad x = \frac{5\pi}{18} + \frac{2}{3}k\pi.$$

Therefore, the solutions in the interval $[0,2\pi)$ that satisfy the original equation are $x = \frac{\pi}{18}, \frac{13\pi}{18}, \frac{25\pi}{18}, \frac{5\pi}{18}, \frac{17\pi}{18}, \frac{29\pi}{18}$.

(10%) **(ii)** Solve the inequality $|2x|+|x-3| < 5$ and express your answer as a union of intervals.

We have three cases to consider: $x<0$, $0\le x<3$, and $x\ge 3$.

If $x<0$, then

$$|2x|+|x-3| = -2x+(3-x) = -3x+3 < 5 \implies -3x < 2 \implies x > -\frac{2}{3},$$

but $x<0$, so the solution set in this case is $x\in(-\frac{2}{3},0)$.

If $0\le x<3$, then

$$|2x|+|x-3| = 2x+(3-x) = x+3 < 5 \implies x < 2,$$

but $0\le x<3$, so the solution set for this case is $x\in[0,2)$.

If $x \geq 3$, then

$$|2x| + |x-3| = 2x + (x-3) = 3x - 3 < 5 \implies 3x < 8 \implies x < \frac{8}{3},$$

but this does not satisfy the condition that $x \geq 3$, so there are no solutions in this case.
Hence, the set of solutions that satisfy the inequality is $x \in (-\frac{2}{3}, 2)$.

(12%) **3.** Recall that $\{F_n\}$, the Fibonacci sequence, is defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Let r be a positive constant which satisfies the property that $r^2 = r + 1$.

Prove for all $n \geq 2$ that

$$r^n = F_n r + F_{n-1}.$$

The proof is by induction. If $n = 2$, we have $r^2 = F_2 r + F_1$, where $F_1 = F_2 = 1$, so the statement is true for the base case.

Now assume that the statement is true when $n = k$, in other words, assume $r^k = F_k r + F_{k-1}$. We need to show the statement is true for $n = k+1$, i.e.

$$\begin{aligned} r^{k+1} &= r(r^k) \\ &= r(F_k r + F_{k-1}) \qquad \text{(by the induction hypothesis)} \\ &= r^2 F_k + r F_{k-1} \\ &= (r+1)F_k + rF_{k-1} = r(F_k + F_{k-1}) + F_k \\ &= rF_{k+1} + F_k \qquad \text{(by the definition of the Fibonacci sequence)}, \end{aligned}$$

so the statement is true for all $n \geq 2$.

4.

(5%) **(a)** State the precise definition of the statement: $\lim_{x \to a} f(x) = L$.

For any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$.

(10%) **(b)** Prove, using the precise definition, $\lim_{x \to \frac{1}{2}} \frac{8x^2}{x-1} = -4$.

We need to show for any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x - \frac{1}{2}| < \delta$ implies $|\frac{8x^2}{x-1} + 4| < \varepsilon$. It is sufficient to choose $\delta = \min(\frac{1}{4}, \frac{\varepsilon}{56})$. If $0 < |x - \frac{1}{2}| < \delta$, then $\delta \leq \frac{1}{4}$ (which we choose so that we stay away from $x = 1$) and $\delta \leq \frac{\varepsilon}{56}$. Then

$$|x - \tfrac{1}{2}| < \tfrac{1}{4} \implies \tfrac{1}{4} < x < \tfrac{3}{4} \implies |x+1| < \tfrac{7}{4}.$$

Furthermore,

$$\tfrac{1}{4} < x < \tfrac{5}{4} \implies -\tfrac{3}{4} < x - 1 < -\tfrac{1}{4} \implies \tfrac{1}{4} < |x-1| < \tfrac{3}{4} \implies \tfrac{1}{|x-1|} < 4.$$

Thus,

$$\begin{aligned} \left|\frac{8x^2}{x-1} + 4\right| = \left|\frac{8x^2 + 4(x-1)}{x-1}\right| = \left|\frac{8x^2 + 4x - 4}{x-1}\right| = 4\left|\frac{(2x-1)(x+1)}{x-1}\right| \\ = 8|x+1| \cdot \frac{1}{|x-1|} \cdot |x - \tfrac{1}{2}| < 8 \cdot \frac{7}{4} \cdot 4 \cdot \delta = 56\delta \leq 56 \cdot \frac{\varepsilon}{56} = \varepsilon, \end{aligned}$$

so $|\frac{8x^2}{x-1} - (-4)| < \varepsilon$, as required.

(6%) **(c)** Prove, using the precise definition, that $\lim_{x\to 5} \dfrac{1}{x-5}$ does not exist.

We wish to show that is NOT true that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-5| < \delta$ implies $|\frac{1}{x-5} - L| < \varepsilon$. But given any $\varepsilon > 0$, no matter what δ you choose, there is some x satisfying $\frac{1}{x-5} > |L| + \varepsilon$; namely, $x = \min(5+\delta, 5+\frac{1}{|L|+\varepsilon})$, since

$$\frac{1}{x-5} > |L| + \varepsilon \iff x - 5 < \frac{1}{|L|+\varepsilon} \iff x < 5 + \frac{1}{|L|+\varepsilon}.$$

But this x does NOT satisfy $|\frac{1}{x-5} - L| < \varepsilon$ since

$$\left|\frac{1}{x-5} - L\right| < \varepsilon \iff L - \varepsilon < \frac{1}{x-5} < L + \varepsilon.$$

5.

(8%) **(i)** Suppose f, g, and h are functions such that $f(x) \le g(x) \le h(x)$ for all real numbers x, f is continuous at a, and h is continuous at a.

If $f(a) = h(a)$, prove that g is continuous at a.

Notice that $f(a) \le g(a) \le h(a)$. Since $f(a) = h(a)$, it follows that $g(a) = f(a) = h(a)$.

We also have by the continuity of f and h at a that $f(a) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = h(a)$, so by the squeeze (pinching) theorem, it follows that $\lim_{x\to a} g(x) = f(a) = g(a)$. Therefore, by the definition, g is continuous at a.

(5%) **(ii)** Does there exist a function $F(x)$ defined for all real numbers such that $F(x)$ is not continuous <u>for any</u> a, but $(F \circ F)(x)$ is continuous <u>for all</u> a? Justify your answer.

Such a function exists. Consider the function

$$F(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Then F is not continuous for any value of x, but $F \circ F = F(F(x)) = 1$ for all x, which is continuous for all x.

6.

(8%) **(a)** Show that there exists a real number x which satisfies the equation $\sin x = x - 1$.

Consider the function $f(x) = \sin x - (x-1)$, which is continuous for all $x \in \mathbb{R}$. Since $f(0) = 1 > 0$ and $f(2008\pi) = -2008\pi + 1 < 0$, then by IVT there exists a value $c \in (-2008\pi, 0)$ such that $f(c) = 0$, which is equivalent to $\sin c - c + 1 = 0$, which means $\sin c = c - 1$.

(6%) **(b)** Suppose $f(x)$ is a continuous function on $[a,b]$ and that $f(x)$ is always irrational. Prove that $f(x)$ is a constant function.

We prove by contradiction: suppose f is not constant; i.e. there exist two unique values $x_0, x_1 \in [a,b]$ such that $f(x_0) < f(x_1)$. Between any two numbers $f(x_0)$ and $f(x_1)$ exists a rational number t by the density of the rationals. Since f is continuous, by IVT there exists a value c either within the interval (x_0, x_1) (if $x_0 < x_1$) or (x_1, x_0) such that $f(c) = t \in \mathbb{Q}$, which contradicts the assumption that $f(x)$ is always irrational. Therefore f must be a constant function.