MAT 137Y, 2006–2007 Winter Session, Solutions to Term Test 1

1. Evaluate the following limits. Do not use L'Hôpital's Rule to evaluate the limit.

(7%) **(i)** $\lim_{x\to 0} \frac{(x-1)^2-1}{x^2+6x}$

$$L = \lim_{x\to 0} \frac{x^2-2x+1-1}{x(x+6)} = \lim_{x\to 0} \frac{x^2-2x}{x(x+6)} = \lim_{x\to 0} \frac{x(x-2)}{x(x+6)} = \lim_{x\to 0} \frac{x-2}{x+6} = -\frac{1}{3}.$$

(7%) **(ii)** $\lim_{t\to 0} \frac{\sin^2(5t)}{3t^2}$

Using the property that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and (as a result of making the substitution $x = 5t$) we have $\lim_{t\to 0} \frac{\sin 5t}{5t} = 1$,

$$\lim_{t\to 0} \frac{\sin^2(5t)}{3t^2} = \frac{1}{3} \lim_{t\to 0} \frac{\sin 5t}{t} \cdot \frac{\sin 5t}{t} = \frac{25}{3} \lim_{t\to 0} \frac{\sin 5t}{5t} \cdot \frac{\sin 5t}{5t} = \frac{25}{3} \cdot 1 \cdot 1 = \frac{25}{3}.$$

(7%) **(iii)** $\lim_{x\to 4^+} \frac{(4-x)|3x-14|}{|4-x|}$.

If $x \to 4^+$, then $x > 4$, so $|4-x| = |x-4| = x-4$. Therefore,

$$\lim_{x\to 4^+} \frac{(4-x)|3x-14|}{|4-x|} = \lim_{x\to 4^+} \frac{(4-x)|3x-14|}{x-4} = \lim_{x\to 4^+} -|3x-14| = -2.$$

(7%) **(iv)** $\lim_{x\to 0} \frac{3-\sqrt{9-x^2}}{x^2}$.

Multiplying top and bottom by the conjugate, we have

$$\lim_{x\to 0} \frac{3-\sqrt{9-x^2}}{x^2} = \lim_{x\to 0} \frac{(3-\sqrt{9-x^2})(3+\sqrt{9-x^2})}{x^2(3+\sqrt{9-x^2})} = \lim_{x\to 0} \frac{9-(9-x^2)}{x^2(3+\sqrt{9-x^2})}$$
$$= \lim_{x\to 0} \frac{x^2}{x^2(3+\sqrt{9-x^2})} = \lim_{x\to 0} \frac{1}{3+\sqrt{9-x^2}} = \frac{1}{6}.$$

2.

(7%) **(i)** Solve the inequality $\frac{x^2-3x}{x^4-1} \le 0$. Express your answer as a union of intervals.

Factoring the expression, we are asked to solve the inequality $\frac{x(x-3)}{(x^2+1)(x+1)(x-1)} \le 0$. The expression is zero at $x = 0, 3$ and undefined at $x = \pm 1$. To find where the expression is positive or negative, we analyze the sign of each of the factors using a number chart:

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < 3$	$x > 3$
x	$-$	$-$	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	$-$	$+$
x^2+1	$+$	$+$	$+$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$	$+$
$\frac{x(x-3)}{(x^2+1)(x+1)(x-1)}$	$+$	$-$	$+$	$-$	$+$

So the inequality holds for $x \in (-1, 0] \cup (1, 3]$.

(ii) Suppose $\sin x = \frac{3}{4}$ and $\frac{\pi}{2} \leq x \leq \pi$.
Find the exact value of each of the following expressions.

(6%) **(a)** $\tan x$.
$\tan x = \frac{\sin x}{\cos x}$. To find $\cos x$, we can either draw a triangle or use the identity

$$\sin^2 x + \cos^2 x = 1 \implies \cos x = \pm\sqrt{1-\sin^2 x} = \pm\sqrt{1-(\tfrac{3}{4})^2} = \pm\sqrt{\tfrac{7}{16}} = \pm\tfrac{\sqrt{7}}{4}.$$

However, x is in the second quadrant, so $\cos x < 0$. Thus

$$\tan x = \frac{\sin x}{\cos x} = \frac{3/4}{-\sqrt{7}/4} = -\frac{3}{\sqrt{7}} = -\frac{3\sqrt{7}}{7}.$$

(4%) **(b)** $\cos 2x$.
There are many ways to do this, as there are three different trigonometric identities one can use; here we will use

$$\cos 2x = \cos^2 x - \sin^2 x = \left(\sqrt{\tfrac{7}{16}}\right)^2 - \left(\tfrac{3}{4}\right)^2 = \tfrac{7}{16} - \tfrac{9}{16} = -\tfrac{2}{16} = -\tfrac{1}{8}.$$

3.

(5%) **(a)** Give the precise ε, δ definition of the following statement: $\lim_{x\to a} f(x) = L$.
For every $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x)-L| < \varepsilon$.

(12%) **(b)** Prove that $\lim_{x\to 3} \dfrac{x^2+1}{1-x} = -5$ directly using the precise definition of limit.
We need to show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x-3| < \delta$ implies that $\left|\frac{x^2+1}{1-x} - (-5)\right| < \varepsilon$. We consider

$$\left|\frac{x^2+1}{1-x} - (-5)\right| = \left|\frac{x^2+1}{1-x} + 5\right| = \left|\frac{x^2+1}{1-x} + \frac{5(1-x)}{1-x}\right| = \left|\frac{x^2-5x+6}{1-x}\right| = |x-3|\left|\frac{x-2}{1-x}\right|$$

Suppose we restrict the value of δ so that $\delta \leq 1$. Then $0 < |x-3| < \delta \leq 1 \implies |x-3| < 1 \implies 2 < x < 4$. Then $|x-2| < 2$ and $1/|1-x| < \frac{1}{|-1|} = 1$. Thus

$$\left|\frac{x-2}{1-x}\right| = |x-2| \cdot \frac{1}{|1-x|} < 2 \cdot 1 = 2,$$

so

$$\left|\frac{x^2+1}{1-x} + 5\right| < 2\delta < \varepsilon$$

if we choose $\delta = \min(1, \frac{\varepsilon}{2})$.

4. Consider the sequence of numbers

$$x_1 = \sqrt{1}, \quad x_2 = \sqrt{1+\sqrt{1}}, \quad x_3 = \sqrt{1+\sqrt{1+\sqrt{1}}}, \quad x_4 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}, \ldots,$$

so x_n contains n nested radicals and exactly n ones.

(3%) **(a)** Express x_n in terms of x_{n-1}. $x_n = \sqrt{1+x_{n-1}}$.

(9%) **(b)** Prove for all positive integers $n \geq 2$ that x_n is irrational.

We prove that x_n is irrational for all $n \geq 2$ by *induction*. The statement is true for the base case $n = 2$ since $x_2 = \sqrt{2}$ is irrational. Now suppose x_{k-1} is irrational; we prove that x_k is also irrational. To prove this, suppose the contrary; that x_k is rational. Then by part (a),

$$x_k = \sqrt{1+x_{k-1}} \implies x_k^2 = 1 + x_{k-1} \implies x_{k-1} = x_k^2 - 1$$

Since x_k is assumed to be rational, then x_k^2 must also be rational, and thus x_{k-1} is merely the difference of two rational numbers, thereby implying that x_{k-1} is also rational. This contradicts our induction hypothesis that x_{k-1} is irrational. Therefore, x_k must be irrational, and so x_n is irrational for all integers $n \geq 2$.

5. For each of the following statements determine whether the statement is true or false (by putting a checkmark next to the word "True" or "False") and then justifying your answer with either a proof showing that it is true, or an example showing that it is false. Failing to check either True or False for each question will result in no credit (and no part marks).

(5%) **(i)** If $g(x)$ is odd and $h(x) = (f \circ g)(x)$, then $h(x)$ is also odd. True ☐ or False ☑

Let $f(x) = x^2$ and $g(x) = x^3$. Then $h(x) = f(g(x)) = f(x^3) = x^6$, which is an even function and not odd.

(5%) **(ii)** If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist,
then $\lim_{x\to a}\big(f(x)+g(x)\big)$ also does not exist. True ☐ or False ☑

Let $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$. Then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ both do not exist, but
$\lim_{x\to 0} f(x) + g(x) = 0$.

(5%) **(iii)** If $f(x) = x^3 - x^2 + x$, then there exists c such that $f(c) = 15$. True ☑ or False ☐

Since f is a polynomial, f is continuous everywhere. Since $f(0) = 0 < 15$ and $f(3) = 27 - 9 + 3 = 21 > 15$, then by the Intermediate Value Theorem there exists $c \in (0,3)$ such that $f(c) = 15$.

(5%) **(iv)** Suppose S is a set of real numbers and lub S exists.
Then S has a largest element. True ☐ or False ☑

Let $S = \{x \mid 0 < x < 1\}$. Then the least upper bound of S is 1, but the interval $(0,1)$ has no largest element.

(6%) **6.** Suppose f is continuous at a and $f(a) > 1$. Prove that there exists $\delta > 0$ such that $f(x) > 1$ for all $x \in (a-\delta, a+\delta)$.

Since f is continuous at a, then $\lim_{x\to a} f(x) = f(a)$, or

$$\text{for any } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that } 0 < |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

In particular, let $\varepsilon = f(a) - 1 > 0$. Then there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x)-f(a)| < f(a) - 1 \implies f(a) - (f(a)-1) < f(x) < f(a) + (f(a)-1),$$

so $1 < f(x)$ for all $|x-a| < \delta$ or $x \in (a-\delta, a+\delta)$, as required.