MAT 137Y, Limit Proofs and Some Examples

We will do our first example at great length with much commentary and explanation. You should not regard it as a model for your proofs but as a guide to thinking through your solution.

Example 1. Find a number $\delta > 0$ such that $|x-2| < \delta$ implies $|x^3 - 8| < 10^{-4}$. In other words find a $\delta > 0$ such that whenever x satisfies $|x-2| < \delta$ then it also satisfies $|x^3 - 8| < 10^{-4}$.

First Solution. We have to make $|x^3 - 8|$ small (specifically, less than 10^{-4}) by making x "sufficiently close" to 2, namely by making $|x-2|$ "sufficiently small". Imagine that someone (Peggy, say) has challenged us to make $|x^3 - 8|$ small and we get to decide how small we want to make $|x-2|$ in order to achieve this. The smallness of $|x-2|$ is controlled by the number δ which we are going to choose. We are going to choose this δ and then prove to Peggy that it works, that is that $|x-2| < \delta$ implies that $|x^3 - 8| < 10^{-4}$. (To put it another way we are going to show her that no matter what value of x she cares to take in the interval $(2-\delta, 2+\delta)$ then, for that value of x, $f(x)$ will lie in the interval $(8-10^{-4}, 8+10^{-4})$.

It is essential to understand that there is no one "correct" value for δ. True, there is theoretically a largest possible value of δ which will work but it will generally be difficult if not impossible to find. Fortunately we don't have to find it! We are quite free to make δ much smaller than it really needs to be. It may well be that the δ we provide to Peggy in fact makes $x^3 - 8 < 10^{-10}$. If so, so much the better: overkill is certainly not forbidden, and in fact makes life easier. Speaking of overkill it is also worth noting that if we find a value δ which works then any smaller value of δ, for example $\delta/2$, will also work (why?). Having made these preliminary comments let's get to work.

What we are going to do is work with the expression $|x^3 - 8|$ to see how we can make it less than 10^{-4}. Our work will basically be a long chain of inequalities $|x^3 - 8| < \ldots\ldots < 10^{-4}$. Some of these strict inequalities may be replaced by "less than or equal to" or "equal to": as long as we have at least one strict inequality in our chain we can conclude that the first element of the chain is less than the last. At each step we will have to justify the inequality. Sometimes the inequality will hold without any assumptions on x (for example if we apply the triangle inequality). Sometimes the inequality will follow from a specific assumption we make about how small $|x-2|$ is (for example $|x-2| < \frac{1}{25}$ or $|x-2| < 1$). We are free to make as many such assumptions as we like since the smallness of $|x-2|$ is completely under our control.

Since $|x-2|$ is what we can control we need to see it in $|x^3 - 8|$ in order to get started:

$$(1) \qquad |x^3 - 8| = |x-2||x^2 + 2x + 4|.$$

The $|x-2|$ factor above we can make small. Can we also make $|x^2 + 2x + 4|$ small? No: since x is going to be close to 2 we expect $|x^2 + 2x + 4|$ to be close to 12. But that's OK: if we can ensure that $|x^2 + 2x + 4|$ is **not too big**, say less than 73, then by making $|x-2| < 10^{-4}/73$ we will be in business. What we need to do is **bound** $|x^2 + 2x + 4|$, that is find some specific number which is larger than $|x^2 + 2x + 4|$. With this aim in mind, continuing from (1), using the triangle inequality and properties of absolute value we have

$$(2) \qquad |x-2||x^2 + 2x + 4| \le |x-2|(|x^2| + |2x| + |4|) = |x-2|(|x|^2 + 2|x| + 4).$$

Now if we just had a bound on $|x|$ we could easily get one on $|x|^2 + 2|x| + 4$. But we can bound $|x|$ since we can make it close to 2. Clearly $|x-2|$ does not need to be particularly small to achieve this but we do need to put some **specific** restriction on $|x-2|$, say $|x-2| < 10$ or $|x-2| < 1$, since $|x-2|$ is what we have control over. The traditional choice is $|x-2| < 1$ (rationale: 1 is the "simplest" strictly positive number). In this example, to highlight the arbitrary nature of this choice we will make the assumption that

$$(a) \qquad |x-2| < 10.$$

Condition (a) implies that $-8 < x < 12$, so $|x| < 12$. So continuing from (2)

$$|x-2|(|x|^2+2|x|+4) < |x-2|(12^2+2\cdot 12+4) = |x-2|(172). \tag{3}$$

Now if we impose the second condition

$$|x-2| < 10^{-4}/172 \tag{b}$$

then

$$172|x-2| < 10^{-4}, \tag{4}$$

which is what we were aiming for. To summarize, (1), (2), (3), and (4) together form a string of inequalities which show that $|x^3-8| < 10^{-4}$ **provided** that our two assumptions (a) and (b) hold. Since $10^{-4}/172 < 10$ we really only need to assume (b) (but we couldn't have foreseen what (b) would be before we imposed (a)). So, we have shown that $|x^3-8| < 10^{-4}$ provided that $|x-2| < 10^{-4}/172$. In other words $\delta = 10^{-4}/172$ is a solution to our problem. ■

Here is an alternate approach to Example 1.

Second solution. Let $x = 2+h$ so $h = x-2$. Then the problem is to find $\delta > 0$ such that $|h| < \delta$ implies $|(2+h)^3-8| < 10^{-4}$. Now

$$\begin{aligned} |(2+h)^3-8| &= |8+12h+6h^2+h^3-8| \qquad \text{(binomial expansion)} \\ &\le 12|h|+6|h|^2+|h|^3 \\ &= |h|(12+6|h|+|h|^2). \end{aligned}$$

Since we can make $|h|$ small we need only bound the other factor, which we do by assuming

$$|h| < 1. \tag{a}$$

Then we have

$$|h|(12+6|h|+|h|^2) < |h|(12+6+1) < 10^{-4},$$

provided we also assume that

$$|h| < 10^{-4}/19. \tag{b}$$

So, under the assumption that (a) and (b) hold, which amounts to the assumption that $|h| < 10^{-4}/19$ we have shown that $|(2+h)^3-8| < 10^{-4}$. This means that $\delta = 10^{-4}/19$ is a solution to the problem. ■

Note: *The second solution is not intrinsically "better" than the first. It is better in that it is shorter but this is only because we have not included the extensive comments. The first solution could easily be pared down to about the same length. It is up to you to decide which approach you like better. Ideally you should understand both.*

It should be clear in Example 1 that we could replace 10^{-4} by any positive number ε, no matter how small, and still come up with a δ which does the job. The work would be very close to the case when $\varepsilon = 10^{-4}$ which we just did. Let's do it explicitly.

Example 2. Given $\varepsilon > 0$ find $\delta > 0$ such that $|x-2| < \delta$ implies $|x^3-8| < \varepsilon$.

From our work in Example 1 we know that if we assume $|x-2| < 10$ then

$$|x^3-8| = |x-2||x^2+2x+4| < 172|x-2|.$$

If we assume further that $|x-2| < \frac{\varepsilon}{172}$ then $172|x-2| < \varepsilon$.

So, we have shown that $|x^3-8| < \varepsilon$ provided $|x-2| < 10$ and $|x-2| < \frac{\varepsilon}{172}$, which means that we want $|x-2|$ to be less than the smaller of these two numbers. Since we don't know what ε is we can't be sure which is the smaller, so we just take δ to be whichever one is smaller, that is $\delta = \min(10, \varepsilon/172)$. (In general $\min(a_1, a_2, \ldots, a_n)$ denotes the smallest of the n numbers in question.) ■

Example 2 is in fact a proof that

$$\lim_{x\to 2} x^3 = 8.$$

Note that the definition of this limit actually says that $0 < |x-2| < \delta$ implies $|x^3-8| < \varepsilon$ (that is x is not allowed to be equal to 2.) This is because in general the function $f(x)$ in question (x^3 in this case) may not be defined at 2, or if it is, it's value may not be equal to the limit. The limit reflects the behaviour of the function near 2 but not at 2. In Example 2 we actually proved more than we needed to, namely even when $x = 2$ we still have $|x^3-8| < \varepsilon$ (in fact $|x^3-8| = 0$). The reason this is possible in this case is that x^3 is a well-behaved function, namely it's value at 2 is the same as it's limit at 2. (As you know, this is what it means for x^3 to be continuous at 2.)

It is only slightly more complicated to prove that as $x \longrightarrow a$, then $x^3 \longrightarrow a^3$.

Example 3. Prove that $\lim_{x\to a} x^3 = a^3$ for any real number a.

We have to show that given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(0) \qquad 0 < |x-a| < \delta \text{ implies } |x^3-a^3| < \varepsilon.$$

We calculate

$$(1) \qquad |x^3-a^3| = |x-a||x^2+ax+a^2| \le |x-a|(|x|^2+|a||x|+|a|^2)$$

In order to bound the second factor above assume

$$(a) \qquad |x-a| < 1.$$

Then $|x| = |x-a+a| \le |x-a| + |a| < 1 + |a|$, so

$$|x|^2+|a||x|+|a|^2 < (1+|a|)^2+|a|(1+|a|)+|a|^2.$$

Let's write C for $(1+|a|)^2+|a|(1+|a|)+|a|^2$. Continuing from (1) we have

$$|x-a|(|x|^2+|a||x|+|a|^2) < |x-a|C < \varepsilon,$$

if we assume that

$$(b) \qquad |x-a| < \varepsilon/C.$$

Noting the assumptions (a) and (b) that we made we see that if $\delta = \min(1, \frac{\varepsilon}{C})$ then (0) is satisfied. ■

Remarks: In Example 2, where $a = 2$, we were able to bound $|x|$ in a very straightforward way. In this example it is slightly more subtle because we don't know whether a is positive or negative. Nonetheless the trick of writing $|x| = |x-a+a|$ and then using the triangle inequality is quite natural since $|x-a|$ is the quantity over which we have control.

Note that our choice of δ in Example 3 is legitimate **because C is a constant**. One mistake that students sometimes make goes something like this

$$|x^3-a^3| = |x-a||x^2+ax+a^2|,$$

so just take $\delta = \varepsilon/|x^2 + ax + a^2|$. This is complete nonsense. The definition of a limit becomes meaningless if you allow δ to depend on x. (Why?) δ may depend on everything in sight (e.g. ε and a) **except for** x. The fact that it may not depend on x is implicit in the definition, since δ appears in the defintion before any mention is made of x.

Example 4. Prove that $\lim_{x\to 2} \frac{1}{x^2} = \frac{1}{4}$.

Given $\varepsilon > 0$ we have to find $\delta > 0$ such that

$$(0) \qquad 0 < |x-2| < \delta \text{ implies } \left|\frac{1}{x^2} - \frac{1}{4}\right| < \varepsilon.$$

Now

$$(1) \qquad \left|\frac{1}{x^2} - \frac{1}{4}\right| = \left|\frac{4-x^2}{4x^2}\right| = \frac{|2-x||2+x|}{4|x|^2} \leq \frac{|x-2|(|2|+|x|)}{4|x|^2}.$$

(Why is $|x-2|$ equal to $|2-x|$?) We will be in good shape if we can bound $\frac{|2|+|x|}{4|x|^2}$. To do this we need an upper bound on the numerator and a lower bound $c > 0$ on the denominator. Assume

$$(a) \qquad |x-2| < 1.$$

Then $1 < x < 3$ so $|x| < 3$ and $|x| > 1$. Continuing with (1) we have

$$\frac{|x-2|(|2|+|x|)}{4|x|^2} < |x-2|\frac{2+3}{4\cdot 1^2} = \frac{5}{4}|x-2|.$$

(We replaced the numerator of the fraction by something larger and the denominator by something smaller, which makes the fraction larger.) Now assuming

$$(b) \qquad |x-2| < \frac{4}{5}\varepsilon$$

we have

$$\frac{5}{4}|x-2| < \varepsilon.$$

Recalling our assumptions (a) and (b) we see that the choice $\delta = \min(1, \frac{4}{5}\varepsilon)$ ensures that (0) holds. ■

Example 5. Prove that $\lim_{x\to\frac{1}{2}} \frac{1}{x^2} = 4$.

Given $\varepsilon > 0$ we must find $\delta > 0$ such that

$$(0) \qquad 0 < \left|x - \frac{1}{2}\right| < \delta \text{ implies } \left|\frac{1}{x^2} - 4\right| < \varepsilon.$$

Now

$$(1) \qquad \left|\frac{1}{x^2} - 4\right| = \left|\frac{1-4x^2}{x^2}\right| = \frac{|1-2x||1+2x|}{|x|^2} \leq 2|x-\tfrac{1}{2}|\frac{1+2|x|}{|x|^2}.$$

Notice that, unlike the previous example, here assuming $|x - \frac{1}{2}| < 1$ will not buy us a lower bound on x since it only implies $-\frac{1}{2} < x < \frac{1}{2}$ so x could be extremely close to 0. In order to keep x "well away from 0" we will assume

$$(a) \qquad \left|x - \frac{1}{2}\right| < \frac{1}{4}.$$

(The $\frac{1}{4}$ above could be replaced by any number strictly less than $\frac{1}{2}$.) Then $\frac{1}{4} < x < \frac{3}{4}$ so $|x| > \frac{1}{4}$ and $|x| < \frac{3}{4}$. Continuing with (1),

$$\frac{2|x-\frac{1}{2}|(1+2|x|)}{|x|^2} < \left|x-\frac{1}{2}\right| \cdot \frac{1+2(\frac{3}{4})}{(\frac{1}{4})^2} = \left|x-\frac{1}{2}\right| 40 < \varepsilon,$$

provided we assume

$$(b) \qquad \left|x-\frac{1}{2}\right| < \frac{\varepsilon}{40}.$$

In view of (a) and (b) we see that if we let $\delta = \min(\frac{1}{4}, \frac{\varepsilon}{40})$ then we will have

$$\left|x-\frac{1}{2}\right| < \delta \text{ implies } \left|\frac{1}{x^2}-4\right| < \varepsilon. \blacksquare$$

In the next example we will replace $\frac{1}{2}$ above by an arbitrary non-zero number a. If a is positive we could handle this much as above, replacing the assumption $|x-\frac{1}{2}| < \frac{1}{4}$ by $|x-a| < \frac{a}{2}$. If a is negative you have to do something a little different. It is instructive to do it this way, but to avoid having to consider cases we will first prove the following variant of the triangle inequality. Although it may seem technical it just says that the distance between two points a and b on the real line is at least as big as the difference between their magnitudes, which should be clear if you draw a picture. Of course this says nothing if the difference between the magnitudes is negative but you can always reverse the roles of a and b to make the difference positive. The proof is very simple.

Theorem. $|a-b| \geq |a|-|b|$
Proof: $|a| = |(a-b)+b| \leq |a-b|+|b|$ so $|a-b| \geq |a|-|b|$. $\blacksquare$

Note that the above fact can also be expressed as

$$|a+b| \geq |a|-|b|,$$

by replacing b with $-b$ (keeping in mind that $|-b| = |b|$).

Example 6. Prove $\lim\limits_{x\to a} \frac{1}{x^2} = \frac{1}{a^2}$ for any number $a \neq 0$.

We are given $\varepsilon > 0$ and must find $\delta > 0$ such that

$$(0) \qquad 0 < |x-a| < \delta \text{ implies } \left|\frac{1}{x^2}-\frac{1}{a^2}\right| < \varepsilon.$$

We have

$$(1) \qquad \left|\frac{1}{x^2}-\frac{1}{a^2}\right| = \left|\frac{a^2-x^2}{a^2x^2}\right| = \frac{|a-x||a+x|}{a^2|x|^2} \leq \frac{|x-a|(|a|+|x|)}{a^2|x|^2}.$$

Now assume

$$(a) \qquad |x-a| < \frac{|a|}{2}.$$

Then $|x| = |(x-a)+a| \geq |a|-|x-a| > |a|-\frac{|a|}{2} = \frac{|a|}{2}$. Also $|x| = |x-a+a| \leq |x-a|+|a| < \frac{|a|}{2}+|a|$ so we have $\frac{|a|}{2} < |x| < \frac{3|a|}{2}$. Using this information on the final expression in (1) we get

$$\frac{|x-a|(|a|+|x|)}{a^2|x|^2} < |x-a|\frac{|a|+\frac{3|a|}{2}}{|a|^2(\frac{|a|}{2})^2} = \frac{10|x-a|}{|a|^3} < \varepsilon,$$

where the last step is justified by the assumption that

$$(b) \qquad |x-a| < \varepsilon\frac{|a|^3}{10}.$$

Looking at (a) and (b) we see that choosing $\delta = \min(\frac{|a|}{2}, \frac{|a|^3\varepsilon}{10})$ will ensure that (0) holds. ■

Now let us look at an example where the stipulation $0 < |x-a|$ in the definition of limit is essential.

Example 7. Find $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$, if it exists.

In this case the expression in question is undefined when $x = 1$ so the stipulation $0 < |x-1|$ is important. Since $x-1 = (\sqrt{x}-1)(\sqrt{x}+1)$, $\frac{x-1}{\sqrt{x}-1}$ can be transformed into the expression $\sqrt{x}+1$, which is defined at x=1 and has the limit 2. Nonetheless $\frac{x-1}{\sqrt{x}-1}$ is not interchangeable with $\sqrt{x}+1$, they simply happen to be equal for all values of x **other than** 1. Since the definition of a limit does not depend on the value at 1 the two expressions do have the same limit at 1. Thus the limit is 2.

Example 8. Prove that $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1} = 2$.

Proof: Given $\varepsilon > 0$ we must find $\delta > 0$ such that

$$(1) \qquad 0 < |x-1| < \delta \Rightarrow \left|\frac{x-1}{\sqrt{x}-1} - 2\right| < \varepsilon.$$

In order to have $\sqrt{x}$ defined we should assume at the outset that

$$(a) \qquad |x-1| < 1,$$

so that $x > 0$. Now

$$\begin{aligned}
\left|\frac{x-1}{\sqrt{x}-1} - 2\right| &= \left|\frac{x-1}{\sqrt{x}-1}\frac{\sqrt{x}+1}{\sqrt{x}+1} - 2\right| \\
&= |\sqrt{x}+1-2| \\
&= |\sqrt{x}-1| \\
&= \left|\frac{x-1}{\sqrt{x}+1}\right| \qquad \text{(multiplying top and bottom by } \sqrt{x}+1) \\
&\le |x-1| \qquad \text{(since } \sqrt{x} > 0) \\
&< \varepsilon,
\end{aligned}$$

where the last inequality is justified by the assumption that

$$(b) \qquad |x-1| < \varepsilon.$$

Thus if we let $\delta = \min(\varepsilon, 1)$ then (1) holds. ■

Example 9. Find $\lim_{x\to 0}\frac{|x|}{x}$, if it exists.

Here again $f(x) = |x|/x$ is not defined when $x = 0$. Note that

$$f(x) = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0. \end{cases}$$

Intuitively this makes it clear that the limit does not exist: it wants to be both 1 and -1 at the same time. It can't be 1 because $f(x) = -1$ for all $x < 0$, which is certainly not close to 1. Similarly it can't be -1. More formally, the right hand limit is 1 and the left hand limit is -1. Since these do not agree the limit (two-sided) does not exist. This is not a formal proof that the limit does not exist but you can try to make your own following the model of Example 11 below.

Example 10. Find $\lim_{x\to 0^+} \sin(\frac{1}{x})$, if it exists.

As x approaches 0 from the right x will take the values $(\pi/2)^{-1}, (2\pi+\pi/2)^{-1}, (4\pi+\pi/2)^{-1}, \ldots$, at which points $\sin(\frac{1}{x})$ will always take the value 1. On the other hand x will also take the values $(2\pi-\pi/2)^{-1}, (4\pi-\pi/2)^{-1}, (6\pi-\pi/2)^{-1}\ldots$ at which points $\sin(\frac{1}{x})$ is always -1. Thus no matter how small an interval $(0,\delta)$ we look at there will be x's in that interval where $\sin(\frac{1}{x})$ is 1 and other x's where $\sin(\frac{1}{x})$ is -1. Thus there cannot be a single number L which $\sin(\frac{1}{x})$ is getting close to, since L would have to be close to both -1 and 1. This should make it intuitively clear that the limit does not exist, even as a right hand limit, and hence not as a two-sided limit either. See S.H.E. Figure 2.1.13 for an idea of the graph of this function. You can see that as x approaches 0 the graph oscillates more and more rapidly between 1 and -1. Now let's make this intuition into a precise proof.

Example 11. Prove $\lim_{x\to 0^+} \sin(\frac{1}{x})$ does not exist.

Proof: The proof will be by contradiction (in fact this is the only way one could possibly prove a statement of this nature). Suppose the limit did exist, so there is a number L such that

$$\lim_{x\to 0^+} \sin(\tfrac{1}{x}) = L.$$

Taking $\varepsilon = \frac{1}{2}$ in the definition of this limit we find that there is a $\delta > 0$ such that $|\sin(\frac{1}{x}) - L| < \frac{1}{2}$ whenever $0 < x < \delta$. Now find an integer $n > 0$ so large that $x_1 = 1/(2n\pi - \pi/2) < \delta$. Then also $x_2 = 1/(2n\pi+\pi/2) < x_1 < \delta$. Thus we can conclude that both $|\sin(\frac{1}{x_1}) - L|$ and $|\sin(\frac{1}{x_2}) - L|$ are less than $\frac{1}{2}$. But x_1 and x_2 were chosen so that $\sin(\frac{1}{x_1}) = -1$ and $\sin(\frac{1}{x_2}) = 1$. Thus we have $|-1-L| < \frac{1}{2}$ and $|1-L| < \frac{1}{2}$, in other words both -1 and 1 lie in the interval $(L-\frac{1}{2}, L+\frac{1}{2})$. Since this interval has length 1 we conclude that the distance from 1 to -1 is less than 1, which is certainly not true. This contradiction shows that our assumption that the limit existed was false, that is the limit does not exist. ■

Remark: Why did we choose $\varepsilon = \frac{1}{2}$? Actually any $\varepsilon \leq 1$ would work for us here since it would lead to the contradiction that the distance from 1 to -1 is less than 2. This is a contradiction but only barely a contradiction, so we used $\varepsilon = \frac{1}{2}$ to make the contradiction starker, thus (hopefully) improving your intuition about this argument. Theoretically, however $\varepsilon = \frac{1}{2}$ is no better than $\varepsilon = 1$, they both do the job.

A careful inspection of the proof of Example 11 reveals that we used the following fact: if R is a positive real number then there is a positive integer n such that $n > R$. (In order to make $1/(2n\pi - \pi/2) < \delta$ we must make $n > \frac{(\delta^{-1}+\frac{\pi}{2})}{2\pi} = R$.) This is called the **Archimedean property** of the real numbers. For purposes of this course we may regard it as obvious and you needn't worry about its name or its proof. For those who are interested, here is a proof. The proof uses the least upper bound axiom which will be introduced later in the course.

Proposition. (Archimedean property of the real numbers) If R is a positive real number then there is a positive integer n such that $n > R$.

Proof: The proof will be by contradiction so we suppose the statement is false, that is $n \leq R$ for all positive integers n. In other words R is an upper bound for $\mathbb{N}$, the set of positive integers. By the least upper bound axiom then $\mathbb{N}$ has a least upper bound b. In particular

$$b \geq n,\ \forall n \in \mathbb{N}. \tag{1}$$

I claim that also

$$b - 1 \geq n, \ \forall n \in \mathbb{N}. \tag{2}$$

The reason that (2) is true is that it is equivalent to the statement that $b \geq n+1$ for all positive integers n, which follows from (1) since $n+1$ is a positive integer whenever n is a positive integer. (2) says that $b-1$ is also an upper bound for $\mathbb{N}$ which contradicts the fact that b is the least upper bound. ■