Department of Mathematics, University of Toronto MAT 137Y, 2009-10 Winter Session	Problem Set Supplement #1 Not to be handed in. Assignment Posted/Revised: October 29, 2009, 16:29

While the following problems do not have to be handed in, you will be responsible for this material for the first term test.

1. SHE 11.1: 1, 3, 5, 7, 11, 13, 21, 23.

2. One property of rational and irrational numbers we have used in our problem sets is that the set of rational and irrational numbers are *dense*; that is, between any two numbers there exist rational and irrational numbers. Let $x, y \in \mathbb{R}$, $r, s \in \mathbb{Q}$, and $t \notin \mathbb{Q}$.

   (a) Suppose x and y are real numbers such that $y - x > 1$. Prove that there exists $k \in \mathbb{Z}$ such that $x < k < y$. Hint: Let l be the largest integer which satisfies $l \leq x$. Consider the integer $l + 1$.

   (b) Given any two numbers x and y such that $x < y$, prove that there exists an integer n such that $y - x > \frac{1}{n}$.

   (c) Using part (b) and then part (a), show that there exists a rational number r such that $x < r < y$, thereby showing the rationals are dense.

   (d) Now, suppose $r < s$, where r and s are rational. Prove there exists an irrational number between r and s by constructing an irrational number t such that $r < t < s$. You may use the fact that $\sqrt{2}$ is irrational.

   (e) Suppose x and y are any real numbers such that $x < y$. Using parts (c) and (d), show there is an irrational number between x and y, thereby showing that the irrationals are dense.

3. For each of the following sets (which are subsets of the real numbers), find (if it exists) the maximum, minimum, least upper bound, and greatest lower bound. Justify your answers with a proof.

   (i) $S = (0, 4]$.

   (ii) $S = \{x \mid x = \frac{(-1)^n}{n},\ n \in \mathbb{N}\}$.

   (iii) $S = \{2^k \mid k \in \mathbb{Z}\}$.

   (iv) $S = \{x \mid x^3 - 6x^2 + 11x - 6 < 0,\ x \notin \mathbb{Q}\}$.

4. Let $S = \{2, 3, 5, 7, 11, 13, \ldots, p_n, \ldots\}$ be the set of all prime numbers. Recall that a positive integer p is prime if it has no divisors other than 1 or p.

   (a) Prove that there are infinitely many primes. (This is considered to be one of the most beautiful proofs in mathematics. If you need a hint, ask your TA in tutorial.)

   (b) Find the least upper bound, greatest lower bound, maximum, and minimum of S, or show they do not exist. Justify your answers.

Note: This is the last handout of problems prior to the first term test. You are responsible for all material that has been covered up to and including this point. Information on the first term test will be posted on the course website.