Math 246S: Homework 8

Due at the beginning of tutorial Tuesday, April 3, 2012 at 8:10 PM sharp.

In this problem set you are allowed to use the following fact (not proved in class): the constructible numbers are all algebraic numbers.

Problems $(1)-(5)$ **are not to be handed in**.

(1) Explain how to construct $\frac{2+\sqrt{5}}{3}$ using ruler and compass.

(2) Let $P(x)$ be a monic cubic polynomial with rational coefficients. (Recall that monic means the leading coefficient is 1.) Suppose $P(x)$ has a complex root of the form $a+bi$ where both a and b are rational. Prove that $P(x)$ has a rational root. Hint: Show that the sum of the 3 roots is rational. Also show that the complex conjugate of $a+bi$ must also be a root.

(3) Show that if $sin\,\alpha$ and $sin\,\beta$ are constructible then $sin\,(\alpha+\beta)$ is also constructible.

(4) Prove that $\frac{\sqrt{5}+\sqrt[3]{5}}{5}$ is not constructible.

(5) Find a tower of fields $\mathbb{Q}=F_0\subset F_1\subset F_2\subset F_3$ such that $\sqrt{1+\sqrt{2}+\sqrt{\sqrt{2}}}\in F_3$.

The following problems are to be handed in:

(6) Prove that the following equation has no constructible solutions:

$$x^3-6x+2\sqrt{2}=0.$$

Hint: You could use the theorem concerning roots of cubic polynomials with rational coefficients if you make an appropriate substitution.

(7) For each of the following angles, decide if it is constructible or not. Justify your answer.
   (a) 37.5°
   (b) An angle θ such that $\cos\theta=\frac{\pi}{6}$
   (c) An angle θ such that $\frac{\theta}{3}$ is constructible
   (d) An angle θ such that $\tan\theta=0.1$

(8) For each of the following numbers, decide if it is constructible or not. Justify your answer.
   (a) $\cos\frac{\pi}{4}$

(b) $\sqrt{7+\sqrt{5}}$
(c) $\sqrt[3]{\frac{9}{10}}$
(d) $\sqrt[3]{\frac{\sqrt{2}}{4}}$
(e) $\sqrt{(\sqrt{\pi}+1)^2-(\sqrt{\pi}-1)^2}$
(f) $\frac{\pi^2}{5}$

(9) (a) Is $\{a+b\sqrt[3]{2}\,|\,a,b\in\mathbb{Q}\}$ a number field? Justify your answer.
(b) Recall that a real number is called *algebraic* if it is a root of some polynomial with integer coefficients. A real number is called *transcendental* if it is not algebraic. Let t be a transcendental number. Show that $\{a+bt\,|\,a,b\in\mathbb{Q}\}$ is not a number field.

(10) Let $0<\theta<\pi/2$ be the angle with $cos\,\theta=\frac{2}{7}$. Show that θ is constructible but $\frac{\theta}{3}$ is not.