Math 246S: Homework 7

Due at the beginning of tutorial Tuesday, March 20, 2012 at 8:10 PM sharp.

(1) (a) Let $\mathbb{N} \times \mathbb{N} := \{(n, m) \mid n, m \in \mathbb{N}\}$ be the set of pairs of natural numbers. Show that $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$ by finding a bijective map $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

(b) Let $\mathbb{N}^k := \{(n_1, n_2, \cdots, n_k) \mid n_i \in \mathbb{N}, 1 \leq i \leq k\}$ be the set of k-tuples of natural numbers. Show by induction that $|\mathbb{N}^k| = |\mathbb{N}|$ for all natural numbers k.

(2) Let $S := \{1, 2, 3, \cdots 10^6\}$. Let T be the set of all subsets of S. Prove that there is no one-to-one map $f : T \to S$.

(3) Suppose S and T are two countable sets. Prove that $S \cup T$ is countable.

(4) Let $P(S)$ denote the set of all subsets of S. Suppose S is a finite set. Show that $|P(S)| = 2^{|S|}$. Hint: Let $S = \{s_1, s_2, \cdots, s_n\}$. Represent a subset A of $P(S)$ by a sequence of 0s and 1s of length n such that the i-th element in the sequence is 1 if $s_i \in A$ and is 0 if $s_i \notin A$.

(5) Show directly (without stating any theorems) that $|P(\mathbb{N})| \leq |\mathbb{R}|$. Hint: Represent a subset A of $\mathbb{N}$ by an infinite sequence of 0s and 1s such that the n-th element in the sequence is 1 if $n \in A$ and is 0 if $n \notin A$.

(6) This question is designed to help you digest the proof of the Bernstein-Cantor Theorem.

(a) Let $S = T = \mathbb{N}$, and $f(n) = g(n) = n + 1$. Find the sets $S_\infty, S_S, S_T, T_\infty, T_S, T_T$. Give an explicit formula for the function h (recall h was defined in terms of f and g). Show that h is a bijection.

(b) Let $S = T = \mathbb{Z}$, and $f(n) = g(n) = n + 1$. Find the sets $S_\infty, S_S, S_T, T_\infty, T_S, T_T$. Give an explicit formula for the function h (recall h was defined in terms of f and g). Show that h is a bijection.

(7) Let A be a finite subset of $\mathbb{R}$. Prove that $|\mathbb{R}| = |\mathbb{R} \setminus A|$.

(8) Prove that the set of finite subsets of $\mathbb{N}$ is countable.

(9) A 'book' is a finite sequence of standard characters. The standard characters include all the characters on your computer keyboard. Show that the set of all possible books is countable.

(10) Let A be a countable subset of $\mathbb{R}$.

(a) Prove that $|\mathbb{R} \setminus A|$ is infinite.

(b) Prove that $|\mathbb{R} \setminus A| = |\mathbb{R}|$.

(c) Find the cardinality of the set of transcendental numbers.