1. Let $G$ be a finite group and let $H$ be a proper subgroup.
   (a) Prove that $\bigcup_{g \in G} gHg^{-1} \neq G$.
   In other words, not every element of $G$ is conjugate to an element of $H$.
   (b) Prove that the restriction map $R(G) \to R(H)$ is not surjective. (Here $R(G)$ denotes the space of class functions on $G$.)
   (c) Prove that there exists non-isomorphic representations $V, W$ of $G$ such that $\text{Res}_H^G(V) \cong \text{Res}_H^G(W)$.
   (d) Suppose that we drop the condition that $G$ finite. Find a counterexample to (a).

2. Let $G$ be a finite group and let $V$ be a representation. Consider $\mathbb{C}[G]$ as the left-regular representation.
   (a) Prove that if $v \in V$, then there exists a unique morphism $\phi : \mathbb{C}[G] \to V$ such that $\phi(1) = v$. (This is the universal property of regular representation.)
   (b) Use (a) to prove that if $V$ is irreducible, then the multiplicity of $V$ in $\mathbb{C}[G]$ is $\dim V$. (We proved this in class with characters.)

3. Consider the vector space $V = (\mathbb{C}^m)^\otimes n$. $V$ is an $S_n$-module where the elements in $S_n$ act by permuting the tensor factors. Express $V$ as a direct sum of the modules $M^\mu$.

4. (Exercise 2.12 from Sagan.)
   In this problem, we consider square matrices with rows and columns indexed by partitions of $n$. For writing these matrices, we put a total order on the set of partitions which refines the dual dominance order (so the first row and column in the matrix corresponds to the largest partition $(n)$). For a partitions $\lambda, \mu$, let $S_\lambda$ denote the corresponding Young subgroup of $S_n$, let $K_\mu$ the conjugacy class of permutations of cycle type $\mu$ and let $K_{\lambda\mu}$ denote the Kostka number.
Define two matrices $A, B$ by $A_{\lambda \mu} = |S_{\lambda} \cap K_{\mu}|$ and $B_{\lambda \mu} = |S_{\mu}|K_{\lambda \mu}$. Show that $A, B$ are upper triangular and that $B(A^t)^{-1}$ is the character table of $S_n$. (Hint: use Frobenius reciprocity.) Use this method to find the character table of $S_4$.


Let $A_n$ denote the set of all standard Young tableaux of shape $(n, n)$ and let $C_n = |A_n|$. $C_n$ is called the $n$th Catalan number.

(a) Find bijections between $A_n$ and the following sets.
   i. Sequences $(a_1, \ldots, a_{2n})$ of 0s and 1s such that in each prefix $(a_1, \ldots, a_k)$ there are at least as many 0s as 1s.
   ii. Lattice paths from $(0,0)$ to $(n,n)$ which take steps either right or up and never go above the line $y = x$.
   iii. Triangulations of a convex $(n+2)$-gon using diagonals.

(b) Prove that $C_{n+1} = C_nC_0 + C_{n-1}C_1 + \cdots + C_0C_n$, where by convention $C_0 = 1$.

(c) Prove that $C_n = \frac{1}{n+1} \left( \begin{array}{c} 2n \\ n \end{array} \right)$ using the hook length formula (Theorem 3.10.2).

(d) (Optional) Prove the same expression for $C_n$ starting with (b) and using generating functions.

(e) (Optional) Give a more combinatorial/bijective proof of this expression.