10 The Duistermaat-Heckman Theorem and Cohomology of Symplectic Quotients

10.1 Introduction

Question: If (M, ω) is a symplectic manifold equipped with a Hamiltonian torus action with moment map $\Phi : M \to \mathbf{t}^*$ and η_0 is a regular value in $\mathbf{t}^*$ so $M_{\eta_0} = \Phi^{-1}(\eta_0)/T$ is the symplectic quotient at η_0, how does M_η vary as η varies in a neighbourhood of η_0 consisting of regular values of Φ?

Proposition 10.1 *The critical values of Φ are of the form $\Phi(M^{T'})$ where $M^{T'}$ is the fixed point set of a 1-parameter subgroup $T' \cong U(1)$ of T.*

For topological reasons, only finitely many such subgroups will appear. The images of $\Phi(A)$ where A is a component of M^T, are subsets of intersections of $\Phi(M)$ with hypersurfaces. These hypersurfaces are normal to u_j where $u_j \in \mathbf{t}^*$ generates T'.

Example 10.2 *If $K = SU(3)$ and T is its maximal torus, the coadjoint orbit $\mathcal{O}_\lambda$ (for generic λ) is a Hamiltonian T space. The fixed point set of the T action is the collection of points $w\lambda$ where $w \in W$. The moment map image $\Phi_T(\mathcal{O}_\lambda)$ is a hexagon, the convex hull of $\{w\lambda\}$. See [19] Figures 4.3 and 4.7.*

We shall see

Theorem 10.3 (Duistermaat-Heckman) *If η_0 is a regular value of Φ, then for η in a sufficiently small neighbourhood U of η_0, $M_\eta \cong M_{\eta_0}$ (the two are diffeomorphic). However M_η is not symplectically diffeomorphic to M_{η_0}. In fact, identifying the symplectic forms ω_η on M_η (via the diffeomorphisms) with symplectic forms on M_{η_0}, we have*

$$\omega_\eta = \omega_{\eta_0} + < \eta - \eta_0, c >$$

where $c \in \Omega^2(M_{\eta_0}) \otimes \mathbf{t}$ is a closed differential form. (Informally: the symplectic form on a reduced space M_η depends linearly on η.)

Corollary 10.4 *The symplectic volume* $\mathrm{vol}(M_\eta)$ *of a family of symplectic quotients is a polynomial function of η in a sufficiently small neighbourhood of a regular value η_0.*

We shall prove this theorem starting from

Proposition 10.5 *(Normal form theorem): we give a normal form for the T action, the symplectic structure and the moment map in a neighbourhood of $\Phi^{-1}(\eta_0)$ for any regular value η_0 of Φ (analogous to Darboux theorem).*

10.2 Slices in moment polytope

Lemma 10.6 *If (M, ω) has a Hamiltonian T action and $H \leq T$, then the moment polytope for Φ_H is obtained as follows. By reduction in stages, if $\zeta \in \mathbf{h}^*$ then $M_\zeta := \Phi_H^{-1}(\zeta)/H$ are a family of symplectic manifolds with Hamiltonian action of T/H. The moment polytopes are $\Phi_{T/H}(M_\zeta) = \{\xi \in \Phi_T(M) : \pi_H(\xi) = \zeta\}$.*

Example 10.7

$$M = \mathbb{C}P^2$$

$\Phi_{T/H}(M_\zeta)$ are a family of intervals of length $1 - \zeta$. Thus M_ζ is a 2-sphere with symplectic area $1 - \zeta$ (in other words the symplectic form on M_ζ varies linearly with ζ, as stated in the Duistermaat-Heckman theorem).

10.3 Normal form theorem

(GS, STP Proposition 40.1)

Connections on principal bundles

Lemma 10.8 *If P is a manifold with a free action of G then $P \to M = P/G$ inherits a structure of principal G-bundle. A connection on G is a 1-form*

$$\theta \in \Omega^1(P) \otimes \mathbf{t}$$

for which

(a) $(R_g)^*\theta = \mathrm{Ad}(g^{-1})\theta$

(b) $\theta(X^\#) = X$ for any $X \in \mathbf{g}$.

Example 10.9 *If $U(1) \to P \to M$ is a principal $U(1)$-bundle, then a connection is a 1-form θ for which*

(a) $(R_g)^*\theta = \theta$ (θ is invariant under the $U(1)$ action)

(b) $\theta(X^\#) = X$ for any $X \in i\mathbb{R} := \mathrm{Lie}(U(1))$.

Example 10.10 *If $U(1)^n \to P \to M$ is a principal $U(1)^n$ bundle, then a connection is a collection of 1-forms $(\theta_1, \ldots, \theta_n)$ on P invariant under the action of $T = U(1)^n$ and for which $\theta_j(\xi_k^\#) = \delta_{jk}$ if $\xi_k^\#$ is the vector field generated by the k-th copy of $U(1)$.*

Theorem 10.11 *Let (M, ω) be a symplectic manifold equipped with a Hamiltonian $U(1)$ action. Use $(\theta_1, \ldots, \theta_n)$ to define a symplectic structure on $\Phi^{-1}(\eta_0) \times \mathbb{R}$: Let ω_{η_0} be the symplectic form on $M_{\eta_0} = \Phi^{-1}(\eta_0)/T$ and define a symplectic structure on $\Phi^{-1}(\eta_0) \times \mathbb{R}^n$ by*

$$\omega = \pi^*\omega_{\eta^0} - d(\sum_{j=1}^n t_j\theta_j)$$

where t_j are coordinates on $\mathbb{R}^n \cong \mathbf{t}^*$ corresponding to the coordinates on $\mathbf{t}$ which specify the θ_j.

The action of T is defined by the action on $\Phi^{-1}(\eta_0)$. Then there is a symplectomorphism from a tubular neighbourhood of $\Phi^{-1}(\eta_0)$ in M to a tubular neighbourhood of $\Phi^{-1}(\eta_0) \times \{0\}$ in $\Phi^{-1}(\eta_0) \times \mathbb{R}^n$.

Special case $n = 1$: $U(1)$ action $\Phi^{-1}(\eta_0) \times \{0\}$ in $\Phi^{-1}(\eta_0) \times \mathbb{R}$

$$\omega = \pi^* \omega_{\eta^0} - d(t\theta)$$

Lemma 10.12 *With the symplectic form ω on $\Phi^{-1}(\eta_0) \times \mathbb{R}^n$, the moment map is*

$$\Phi' : (p, (t_1, \ldots, t_n)) \mapsto -(t_1, \ldots, t_n)$$

Proof: $i_{\zeta_j} d(\sum_k t_k \theta_k) = -d i_{\zeta_j}(\sum_k t_k \theta_k) = -dt_j$ (since $L_{\zeta_j} \theta_k = 0$). □

Remark 10.13 *The isomorphism with a tubular neighbourhood is not canonical. It depends on the choice of a connection $(\theta_1, \ldots, \theta_n)$.*

Remark 10.14 *If 0 is a regular value of the moment map, an analogous statement is true for the normal form for the action of a nonabelian group.*

10.4 Duistermaat-Heckman theorem

Theorem 10.15 *If η_0 is a regular value of $\Phi : M \to \mathbf{t}^*$, then for η in a sufficiently small neighbourhood of η_0, $M_\eta \cong M_{\eta_0}$ and*

$$\omega_\eta = \omega_{\eta_0} + \sum_{j=1}^{n} (\eta - \eta_0)_j d\theta_j$$

Here we have decomposed $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n = \mathbf{t}^*$. In other words the symplectic form varies linearly.

Proof: If $\Phi^{-1}(\eta) \times \{\eta\}$ is in the open neighbourhood of $\Phi^{-1}(\eta^0) \times \{\eta^0\}$ which is identified diffeomorphically with a tubular neighbourhood of $\Phi^{-1}(\eta^0)$ in M, then

$$M_\eta = \Phi^{-1}(\eta)/T = (\Phi^{-1}(\eta^0) \times \{\eta\})/T = \Phi^{-1}(\eta^0)/T \times \{\eta\} = M_{\eta^0}.$$

The symplectic form on M_η pulls back on $\Phi^{-1}(\eta^0) \times \{\eta\}$ to the restriction

$$\omega_{\eta^0} - d\left(\sum_j (\eta_j - \eta_j^0)\theta_j\right).$$

But now η is a constant so the symplectic form pulls back to

$$\omega_{\eta^0} - \sum_j (\eta_j - \eta_j^0) d\theta_j.$$

Now $c_j := d\theta_j$ is a closed 2-form on $\Phi^{-1}(\eta_0) \times \mathbb{R}^n$. In fact c_j pulls back from M_{η^0}. It is a representative in de Rham cohomology for the first Chern class of a line bundle L_j over M_{η^0}. If v_j is the element of $\mathbf{t}^*$ defining the coordinate t_j on $\mathbb{R}^n$, take $\rho_j = \exp(v_j) \in \Lambda^W = \mathrm{Hom}(T, U(1))$

$$L_j = \Phi^{-1}(\eta^0) \times_{T,\rho_j} \mathbb{C}$$

is a line bundle over M_{η_0}. Then θ_j gives a connection on L_j so $d\theta_j$ is its curvature.

□

Proposition 10.16 *The pushforward $\Phi_*(\omega^N/N!)$ at $\eta \in \mathbf{t}^*$ is equal to the symplectic volume of M_η multiplied by* $\mathrm{vol}(T)$.

Proof: For a smooth function f on $\mathbf{t}$,

$$\int_{\eta\in\mathbf{t}^*} \Phi_*(\frac{\omega^N}{N!})f(\eta) = \int_{m\in M} \frac{\omega^N}{N!} f(\Phi(m)) = \int_M (\exp\omega) f(\Phi(m)).$$

Choose f supported on the neighbourhood $U \in \mathbf{t}^*$. Then the integral becomes

$$\int_{(p,\eta)\in\Phi^{-1}(\eta^0)\times U} \exp\left(\pi^*\omega_{\eta_0} - d(\eta-\eta_0,\theta)\right) g(\eta)$$

$$= \int_{(p,\eta)\in\Phi^{-1}(\eta^0)\times\mathbf{t}^*} \exp(\pi^*\omega_{\eta_0}) \exp\left(-(d\eta,\theta) - (\eta-\eta_0, d\theta)\right) g(\eta)$$

□ The measure on $\mathbf{t}^*$ comes from $d\eta_1 \wedge \ldots \wedge d\eta_n$ in $d\eta_1 \wedge \ldots \wedge d\eta_n \wedge \theta_1 \wedge \ldots \wedge \theta_n$ obtained by expanding $\exp\{-(d\eta,\theta)\}$. We evaluate the integral over $\Phi^{-1}(\eta_0) \times \{\eta\}$, to get

$$\int \theta_1 \wedge \ldots \wedge \theta_n \exp(\pi^*\omega_{\eta_0}) \exp\left(-(\eta-\eta_0), d\theta\right)$$

$$= \mathrm{vol}(M_\eta)\mathrm{vol}(T).$$

(since $\mathrm{vol}(T) = \int_T \theta_1 \wedge \ldots \wedge \theta_n$.) The remaining integral is over $\eta \in \mathbf{t}^*$, so it is $\int_{\eta\in\mathbf{t}^*} \mathrm{vol}(M_\eta)\mathrm{vol}(T) g(\eta)$ so

$$\mathrm{vol}_\omega(M) = \int_{\mathbf{t}^*} \Phi_*(\omega^N/N!) = \mathrm{vol}(T) \int_{\eta\in\mathbf{t}^*} \mathrm{vol}_\omega(M_\eta).$$

This follows by applying the definition of pushforward to $g : \mathbf{t}^* \to \mathbb{R}$ given by $g(x) = 1$.

Corollary 10.17 *If M^{2N} is a toric manifold (acted on effectively by $U(1)^N$) then its symplectic volume is equal to the Euclidean volume of its Newton polytope. (The proof uses the fact that for a toric manifold, the symplectic quotient at η is a point if η is in the image of the moment map, and it is empty otherwise.)*

Notice that $\Phi_*\omega^N/N!$ is supported on the (compact) polytope $\Phi(M)$ but it encodes more information about M than just the polytope.

Proposition 10.18 *$\Phi_*\omega^N/N!$ is a polynomial of degree $\leq N$ on sufficiently small neighbourhoods of regular values of Φ.*

This follows from the Duistermaat-Heckman theorem. This means it is polynomial on any connected component of the set of regular values of Φ, and it (or its derivatives) have discontinuities on the hyperplanes (walls) consisting of critical values of Φ.

Example 10.19 *hexagon $\Phi_*\omega^N/N!$ is a piecewise linear function characterized by*

1. *$\Phi_*\omega^N/N! = 0$ on the boundary of $\Phi(\mathcal{O}_\lambda)$*
2. *On the region adjacent to the boundary, $\Phi_*\omega^N/N!$ is proportional to the Euclidean distance to the boundary*
3. *$\Phi_*\omega^N/N!$ is constant on the interior triangle (the component of the complement of the walls containing the centre of the hexagon)*

10.5 Remarks on torus actions

Proposition 10.20 *The orbits of a Hamiltonian torus action are isotropic.*

Proof: $\omega(X^{\#}{}_m, Y^{\#}{}_m) = 0$ for any $X, Y \in \mathbf{t}$. □

Corollary 10.21 *If M^{2n} is a toric manifold (acted on by $T \cong U(1)^N$) then*

$$M \xrightarrow{\Phi} B$$

has the property that $\Phi^{-1}(b)$ is a Lagrangian submanifold for any regular value b of Φ. In other words $\Phi^{-1}(\mathrm{Int}B) \to B$ is a fibration with Lagrangian fibres isomorphic to $U(1)^N$.

This is a special case of the Arnold-Liouville statement.

10.6 Computation of pushforward of Liouville measure on symplectic vector space

Recall $V \cong \mathbb{C}^N$ a symplectic vector space acted on linearly by a torus T, in other words

$$V \cong \oplus_j \mathbb{C}_{(j)}$$

where $\mathbb{C}_{(j)}$ is acted on by

$$\rho_\beta = \exp(2\pi i\beta) \in \mathrm{Hom}(T, U(1))$$

for

$$\beta \in \Lambda^W = \mathrm{Hom}(\Lambda^I, \mathbb{Z}) \subset \mathbf{t}^*.$$

We saw that the moment map was

$$\Phi(z_1, \ldots, z_N) = -\frac{1}{2}\sum_j |z_j|^2 \beta_j$$

$$\omega = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}^j = \sum_j dx_j \wedge dy_j.$$

Lemma 10.22

$$\Phi_*(\omega^N/N!)(\xi) = H_{\bar{\beta}}(\xi)$$

where $H_{\bar{\beta}}(\xi) = \mathrm{vol}\{(s_1, \ldots, s_N) \in (\mathbb{R}^+)^N : \xi = -\sum_{j=1}^N s_j\beta_j\}$. *Here, the number of equations is* ℓ *(the dimension of* T*) and the number of unknowns is* N. $H_{\bar{\beta}}$ *is piecewise polynomial of degree* $N - \ell$. $H_{\bar{\beta}}(\xi)$ *is the pushforward of Lebesgue measure on* $\mathbb{R}^N$.

For $N = \ell$, $H_{\bar{\beta}}$ *is the characteristic function of the Newton polytope.*

In case $\Psi : \mathbb{C} \to \mathbb{R}$, $z \mapsto \frac{1}{2}|z|^2$, $\Psi_*(dx \wedge dy) = 2\pi ds$ when s is the coordinate on $\mathbb{R}$. This is because $dxdy = rdrd\theta == \frac{1}{2}d(r^2)d\theta$. So

$$\int f(\frac{1}{2}r^2)\frac{1}{2}d(r^2)d\theta = 2\pi \int f(r^2)d(\frac{1}{2}r^2)$$

so

$$\Psi_*(dxdy) = 2\pi ds.$$

So

$$\mu = L \circ (\Psi_1, \ldots, \Psi_N)$$

where

$$\Psi_j : \mathbb{C} \to \mathbb{R}^+$$

is

$$\Psi_j(z) = \frac{1}{2}|z|^2.$$

So

$$\mu_* = L_*((2\pi)^N ds_1 \wedge \ldots ds_N).$$

It is easy to check that

$$L_*(ds_1 \wedge \ldots ds_N)(\xi) = \mathrm{vol}\left((s_1, \ldots, s_N) \in (\mathbb{R}^+)^N : \xi = -\sum_{j=1}^N s_j\beta_j\right).$$

$$D_{\beta_j} = \beta_j(\frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial \xi_N})$$

is a differential operator on $\mathbf{t}^*$. Then

$$\prod_{j=1}^N D_{\beta_j} H_{\bar{\beta}} = \delta(\xi)$$

so $H_{\bar{\beta}}$ is the fundamental solution of a differential equation with support on the cone $-C_{\bar{\beta}}$.

Let M be a symplectic manifold equipped with the Hamiltonian action of a group G. The element $\bar{\omega} \in \Omega^2_G(M)$ defined by

$$\bar{\omega}(X) = \omega + (\mu, X)$$

satisfies $D\bar{\omega} = 0$ and thus defines an element $[\bar{\omega}] \in H^2_G(M)$.

Theorem 10.23 (Duistermaat-Heckman theorem, version II) *Suppose M is a symplectic manifold of dimension $2n$ equipped with the Hamiltonian action of a torus T. Then for generic $X \in \mathbf{t}$, in the notation of Theorem 8.49, we have*

$$\int_M \frac{(i\omega)^n}{n!} e^{i\mu(m)(X)} = \sum_{F \in \mathcal{F}} e^{i\mu(F)(X)} \int_F \frac{e^{i\omega}}{e_F(X)}.$$

Proof: Apply the abelian localization theorem (Theorem 8.49) to the class $\exp i\bar{\omega} \in H^*_G(M)$. (For each component F of the fixed point set of T, the value of $\mu(F)$ is a constant.) □

10.7 Stationary phase approximation

An alternative approach to this version of the Duistermaat-Heckman theorem ("exactness of the stationary phase approximation") is sketched as follows. Assume for simplicity that $T = U(1)$ and that the components F of the fixed point set are isolated points. By the equivariant version of the Darboux-Weinstein theorem [35], we may assume the existence of Darboux coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on a coordinate patch U_F about F, for which $\mu(x_1, y_1, \ldots, x_n, y_n) = \mu(F) - \sum_j \frac{m_j}{2}(x_j^2 + y_j^2)$. Thus the oscillatory integral over U_F tends (if we may replace U_F by $\mathbb{R}^{2n}$) to

$$\int_{m \in M} e^{i\omega} e^{i\mu(m)X} = \int_{\mathbb{R}^{2n}} i^n dx_1 dy_1 \ldots dx_n dy_n e^{i\mu(F)X} e^{-i\sum_j m_j(x_j^2+y_j^2)X/2}. \tag{3}$$

The integral over $\mathbb{R}^{2n}$ is given by a standard Gaussian integral:

$$\int_{\mathbb{R}^{2n}} e^{i\omega} e^{i\mu X} = \frac{(2\pi)^n e^{i\mu(F)X}}{(\prod_j m_j) X^n} := \mathcal{S}_F(X)$$

The *lemma of stationary phase* ([22], Section 33) asserts that the oscillatory integral $\int_{m \in M} e^{i\omega} e^{i\mu(m)X}$ over M has an asymptotic expansion as $X \to \infty$ given by

$$\int_{m \in M} e^{i\omega} e^{i\mu(m)X} = \sum_{F \in \mathcal{F}} \mathcal{S}_F(X)(1 + O(1/X)) + O(X^{-\infty}).$$

The first version of the Duistermaat-Heckman theorem (Theorem **??**) may thus be reformulated as the assertion that *the stationary phase approximation is exact* (in other words the leading order term in the asymptotic expansion gives the answer exactly for any value of X).

10.8 The Kirwan map

Suppose M is a compact symplectic manifold equipped with a Hamiltonian action of a compact Lie group G. Suppose 0 is a regular value of the moment map μ. There is a natural map $\kappa : H_G^*(M) \to H^*(M_{\mathrm{red}})$ defined by

$$\kappa : H_G^*(M) \mapsto H_G^*(Z_0) \cong H^*(M_{\mathrm{red}});$$

where $Z_0 = \mu^{-1}(0)$: it is obviously a ring homomorphism.

Theorem 10.24 *The map κ is surjective.*

The proof of this theorem ([27], 5.4 and 8.10; see also Section 6 of [29]) uses the Morse theory of the "Yang-Mills function" $|\mu|^2 : M \to \mathbb{R}$ to define an equivariant stratification of M by strata S_β which flow under the gradient flow of $-|\mu|^2$ to a critical set C_β of $|\mu|^2$. One shows that the function $|\mu|^2$ is *equivariantly perfect* (i.e. that the Thom-Gysin (long) exact sequence in equivariant cohomology decomposes into short exact sequences, so that one may build up the cohomology as

$$H_G^*(M) \cong H_G^*(\mu^{-1}(0)) \oplus \bigoplus_{\beta \neq 0} H_G^*(S_\beta).$$

Here, the stratification by S_β has a partial order $>$; thus one may define an open dense set $U_\beta = M - \cup_{\gamma > \beta} S_\gamma$ which includes the open dense stratum S_0 of points that flow into $\mu^{-1}(0)$ (note S_0 retracts onto $\mu^{-1}(0)$). The equivariant Thom-Gysin sequence is

$$\cdots \to H_G^{n-2d(\beta)}(S_\beta) \stackrel{i_{\beta *}}{\to} H_G^n(U_\beta) \to H_G^n(U_\beta - S_\beta) \to \dots .$$

To show that the Thom-Gysin sequence splits into short exact sequences, it suffices to know that the maps $(i_\beta)_*$ are injective. Since $i_\beta^*(i_\beta)_*$ is multiplication by the equivariant Euler class e_β of the normal bundle to S_β, injectivity follows because this equivariant Euler class is not a zero divisor (see [27] 5.4 for the proof).

Atiyah and Bott [2] use a similar argument in an infinite dimensional context to define a stratification of the space of all connections $\mathcal{A}(\Sigma)$ on a compact Riemann surface Σ, using the Yang-Mills functional $\int_\Sigma |F_A|^2$ (which is equivariant with respect to the action of the gauge group $\mathcal{G}$): this stratification is used to compute the Betti numbers of $\mathcal{M}(n, d)$.

10.9 Nonabelian localization

Witten in [38] gave a result (the *nonabelian localization principle*) that related intersection pairings on the symplectic quotient M_{red} of a (compact) manifold M to data on M itself. Since $\kappa : H_G^*(M) \to H^*(M_{\mathrm{red}})$ is a surjective ring homomorphism, all intersection pairings are given in the form $\int_{M_{\mathrm{red}}} \kappa(\eta)$ for some $\eta \in H_G^*(M)$.

Witten [38] regards the equivariant cohomology parameter $X \in \mathbf{g}$ as an integration variable, and seeks to compute the asymptotics in $\epsilon > 0$ of

$$\int_{X \in \mathbf{g}} dX e^{-\epsilon |X|^2/2} \int_M \eta(X) e^{i\omega} e^{i(\mu, X)}.$$

He finds that this has an asymptotic expansion as $\epsilon \to 0$ of the form

$$\int_{M_{\mathrm{red}}} e^{\epsilon \Theta} e^{i\omega_{\mathrm{red}}} \kappa(\eta) + O\Big(p(\epsilon^{-1/2}) e^{-\frac{b}{2\epsilon}}\Big) \tag{4}$$

where b is the smallest nonzero critical value of $|\mu|^2$, p is a polynomial, and Θ is a particular element of $H^4(M_{\mathrm{red}})$ (the image $\kappa(\beta)$ of the element $\beta \in H^*_G(M)$ specified by $\beta : X \in \mathbf{g} \mapsto -|X|^2/2$).

10.10 The residue formula

A related result is the *residue formula*, Theorem 8.1 of [23]:

Theorem 10.25 *([23], corrected as in [24])*
Let $\eta \in H^*_G(M)$ induce $\eta_0 \in H^*(X_{\mathrm{red}})$. Then we have

$$\int_{X_{\mathrm{red}}} \kappa(\eta) e^{i\omega_{\mathrm{red}}} = n_0 C^G \mathrm{Res}\left(\mathcal{D}^2(X) \sum_{F \in \mathcal{F}} H^\eta_f(X)[dX]\right), \tag{5}$$

where n_0 is the order of the stabilizer in G of a generic element of $\mu^{-1}(0)$, and the constant C^G is defined by

$$C^G = \frac{(-1)^{s+n_+}}{|W| \mathrm{vol}(T)}. \tag{6}$$

We have introduced $s = \dim G$ and $l = \dim T$; here $n_+ = (s-l)/2$ is the number of positive roots.[3] *Also, $\mathcal{F}$ denotes the set of components of the fixed point set of T, and if F is one of these components then the meromorphic function H^η_f on $\mathbf{t} \otimes \mathbb{C}$ is defined by*

$$H^\eta_f(X) = e^{i\mu(F)(X)} \int_F \frac{i_F^* \eta(X) e^{i\omega}}{e_F(X)} \tag{7}$$

and the polynomial $\mathcal{D} : \mathbf{t} \to \mathbb{R}$ is defined by $\mathcal{D}(X) = \prod_{\gamma > 0} \gamma(X)$, where γ runs over the positive roots of G.

[3]Here, the roots of G are the nonzero weights of its complexified adjoint action. We fix the convention that weights $\beta \in \mathbf{t}^*$ satisfy $\beta \in \mathrm{Hom}(\Lambda^I, \mathbb{Z})$ rather than $\beta \in \mathrm{Hom}(\Lambda^I, 2\pi\mathbb{Z})$ (where $\Lambda^I = \mathrm{Ker}(\exp : \mathbf{t} \to T)$ is the integer lattice). This definition of roots differs by a factor of 2π from the definition used in [23].

The residue map Res is defined on (a subspace of) the meromorphic differential forms on $\mathbf{t} \otimes \mathbb{C}$: its definition depends on some choices, but the sum of the residues over all $F \in \mathcal{F}$ is independent of these choices. When $T = U(1)$ we define the residue on meromorphic functions of the form $e^{i\lambda X}/X^N$ when $\lambda \neq 0$ (for $N \in \mathbb{Z}$) by

$$\mathrm{Res}(\frac{e^{i\lambda X}}{X^N}) = \mathrm{Res}_{X=0}\frac{e^{i\lambda X}}{X^N}, \text{ if } \lambda > 0;$$

$$= 0, \text{ if } \lambda < 0.$$

More generally the residue is specified by certain axioms (see [23], Proposition 8.11), and may be defined as a sum of iterated multivariable residues $\mathrm{Res}_{X_1=\lambda_1} \dots \mathrm{Res}_{X_l=\lambda_l}$ for a suitably chosen basis of $\mathbf{t}$ yielding coordinates $X_1, \dots, X_l$ (see [25]).

The main ingredients in the proof of Theorem **??** are the normal form theorem (Proposition 4.14) and the abelian localization theorem (Theorem 8.49). We outline a proof as follows. First (following S. Martin [31]) we may reduce to symplectic quotients by the action of the maximal torus T:

Proposition 10.26 [31] *We have*

$$\int_{\mu^{-1}(0)/G} \kappa(\eta e^{i\bar{\omega}}) = \frac{1}{|W|}\int_{\mu^{-1}(0)/T} \kappa(\mathcal{D}\eta e^{i\bar{\omega}}) = \frac{(-1)^{n_+}}{|W|}\int_{\mu_T^{-1}(0)/T} \kappa(\mathcal{D}^2\eta e^{i\bar{\omega}}).$$

Thus we need to prove the result only for torus actions. A sketch of the proof when $G = U(1)$ [26] follows: We write

$$\eta = D\Big(\frac{\theta\eta}{d\theta - X}\Big)$$

Suppose 0 is a regular value of μ. Then $\mu^{-1}(\mathbb{R}^+)$ is a manifold with boundary $\mu^{-1}(0)$. One may show that

$$\mathrm{Res}_{X=0}\int_{Z_0}\frac{\theta\eta e^{i\bar{\omega}}}{X - d\theta} = \int_{Z_0/G}\kappa(\eta e^{i\bar{\omega}}). \tag{8}$$

(since in the $U(1)$ case the map κ may be written as

$$\kappa : \eta \mapsto \mathrm{Res}_{X=0} p_* \frac{\theta\eta}{X - d\theta} \tag{9}$$

where $p : Z_0 \to Z_0/G$ is integration over the fibre). Applying the equivariant Stokes' theorem to $\mu^{-1}(\mathbb{R}^+)$ and then taking the residue at $X = 0$ we find that

$$\mathrm{Res}_{X=0}\int_{Z_0}\frac{\theta\eta}{d\theta - X} - \sum_{F\in\mathcal{F}:\mu(F)>0}\mathrm{Res}_{X=0}e^{i\mu(F)X}\int_F \frac{\eta e^{i\omega}}{e_F(X)} = 0, \tag{10}$$

which is exactly the $U(1)$ case of the residue formula.

Nonabelian localization has had two major applications thus far. The first is that the residue formula has been used in [24] to give a proof of formulas for intersection numbers in moduli spaces of vector bundles on Riemann surfaces: some of the background underlying these results is described in Lecture 5. The second is that nonabelian localization underlies some proofs (see e.g. [25], [34]) of a conjecture of Guillemin and Sternberg [20] that "quantization commutes with reduction": in other words that the G invariant part of the quantization (see Section **??**) of a symplectic manifold equipped with a Hamiltonian G action is isomorphic to the quantization of the reduced space M_{red}. For an expository account and references on results about this conjecture, see the recent survey article by Sjamaar [33].

10.11 The residue formula by induction

Guillemin and Kalkman [18] and independently Martin [31] have given an alternative version of the residue formula which uses the one-variable proof inductively:

Theorem 10.27 (Guillemin-Kalkman; Martin) *Suppose M is a symplectic manifold acted on by a torus T in a Hamiltonian fashion, and $\eta \in H_T^*(M)$. Then*

$$\int_{M_{\mathrm{red}}} \kappa(\eta) = \sum_i{}' \int_{(M_i)_{\mathrm{red}}} \kappa_i(\mathrm{Res}_i \eta).$$

Here, M_i is the fixed point set of a one parameter subgroup T_i of T (so that $\mu_T(M_i)$ are critical values of μ_T): it is a symplectic manifold equipped with a Hamiltonian action of T/T_i and the natural map $\kappa_i : H^*_{T/T_i}(M_i) \to H^*((M_i)_{\mathrm{red}})$. The map $\mathrm{Res}_i : H_T^*(M) \to H^*_{T/T_i}(M_i)$ is defined by

$$\mathrm{Res}_i \eta = \mathrm{Res}_{X_i=0}(i^*_{M_i}\eta) \tag{11}$$

where

$$i^*_{M_i}\eta \in H_T^*(M_i) = H^*_{T/T_i}(M_i) \otimes H^*_{T_i}$$

and $X_i \in \mathbf{t}_i^*$ is a basis element for $\mathbf{t}_i^*$.

The sum in Theorem **??** is over those T_i and M_i for which a (generic) ray in $\mathbf{t}^*$ from 0 to the complement of $\mu_T(M)$ intersects $\mu_T(M_i)$. (Different components M_i and groups T_i will contribute depending on the choice of the ray.)