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9 The symplectic structure on coadjoint orbits

Reference: Audin (2nd ed.) §II.3.c

Theorem 9.1 *Suppose $\lambda \in \mathbf{g}^*$ and $\mathcal{O}_\lambda$ is the coadjoint orbit through λ. Then $\mathcal{O}_\lambda$ carries a symplectic structure (Kirillov-Kostant-Souriau)*

Proof: Define a 2-form ω_ξ on $\mathcal{O}_\xi$ by

$$\omega_\xi(\hat{X}, \hat{Y}) = -\xi([X, Y]) \tag{3}$$

where $\hat{X}$ is the vector field on $\mathbf{g}^*$ generated by the action of G. Note that the tangent space to $\mathcal{O}_\lambda$ at λ is $\{[\lambda, X] | X \in \mathbf{g}\}$. □

Remark 9.2 *If we write the above 2-form in terms of $\xi \in \mathbf{g}^*$, it is canonical (it does not depend on a choice of inner product on $\mathbf{g}$). Often it is more convenient to choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{g}$ to identify $\mathbf{g}^*$ with $\mathbf{g}$ (then we can work with the adjoint action rather than the coadjoint action). Once such an inner product has been chosen, we can write the 2-form as*

$$\omega_\lambda(\hat{X}, \hat{Y}) = -\langle \lambda, [X, Y] \rangle.$$

This 2-form depends on the choice of inner product.

Remark 9.3 *There is an inner product on G which is invariant under both left and right multiplication. This inner product gives rise to an inner product on $\mathbf{g}$ which is invariant under the adjoint action of G. For simple Lie algebras, such an inner product on $\mathbf{g}$ is unique up to multiplication by a constant For semisimple Lie algebras, the inner product is unique up to multiplication by constants on all the simple factors. This inner product is called the* Killing form. *See T. Bröcker, T. Tom Dieck,* Representations of Compact Lie Groups*, Springer (GTM).*

Remark 9.4 *Each $\mathcal{O}_\lambda \cong G/Z(\lambda)$ where $Z(\lambda)$ is the centralizer of λ. For most λ, $Z(\lambda)$ is just one maximal torus. It will always include some maximal torus. For instance if $G = U(n)$ and λ is a diagonal matrix with no equal eigenvalues, then the subgroup of G commuting with λ is just T, the diagonal matrices $U(1)^n$. For some λ, $Z(\lambda)$ is larger, for instance if there are some equal eigenvalues.*

Claim:

1. ω is preserved by the action of G.

2. ω is closed

3. ω restricts to a nondegenerate form on the coadjoint orbit $\mathcal{O}_\lambda$

4. The moment map for the action of G on the orbit is the inclusion map $\mathcal{O}_\lambda \to \mathbf{g}^*$

Proof: (of (1)): Let $f = \mathrm{Ad}(g)$ be the diffeomorphism of $\mathcal{O}_\lambda$ associated to an element $g \in G$. Then

$$(f^*\omega)_\lambda(X^\#, Y^\#) = \omega_{f(\lambda)}(f_*X^\#, f_*Y^\#)$$

We would like to prove this equals

$$\omega_\lambda(X^\#, Y^\#).$$

But

$$X^\#{}_\lambda = [X, \lambda]$$

So

$$\mathrm{Ad}(g)_*X^\# = \mathrm{Ad}(g)[X, \lambda] = [\mathrm{Ad}(g)X, \mathrm{Ad}(g)\lambda] = \mathrm{Ad}(g)X^\#.$$

But

$$\begin{aligned} &\langle \mathrm{Ad}(g)\lambda, [\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y]\rangle \\ &= \langle \mathrm{Ad}(g)\lambda, [\mathrm{Ad}(g)[X, Y]\rangle \\ &= \langle \lambda, [X, Y], .\rangle \end{aligned}$$

□

Proof: (of (2)):
To prove

$$d\omega_\lambda(\hat{X}, \hat{Y}, \hat{Z}) = \frac{1}{3}\Big\{\hat{X}(\omega_\lambda(\hat{Y}, \hat{Z})) - \hat{Y}(\omega_\lambda(\hat{X}, \hat{Z})) + \hat{Z}(\omega_\lambda(\hat{X}, \hat{Y}))\Big\} +$$

$$+ \frac{1}{3}\Big\{-\omega([\hat{X}, \hat{Y}], \hat{Z}) + \omega([\hat{X}, \hat{Z}], \hat{Y}) - \omega([\hat{Y}, \hat{Z}], \hat{X})\Big\}$$

Here, $\hat{X}$ denotes the vector field on $\mathbf{g}$ associated to the adjoint action of an element $X \in \mathbf{g}$.

The first bracket involves terms

$$\hat{X}(\omega_\lambda(\hat{Y}, \hat{Z}) = (\hat{X}f)(x)$$

where $x \in \mathbf{g}$ and

$$f(x) = < x, [Y, Z]$$

Then

$$(\hat{X}f)(x) = < [X, x], [Y, Z] >$$

because the vector field

$$\hat{X}(x) = [X, x].$$

This equals

$$-< x, \big[X, [Y, Z] >$$

(by invariance of the inner product under conjugation). Summing over the three terms in the first $\{\cdot, \cdot\}$, the first $\{\cdot, \cdot\}$ is zero by the Jacobi identity.

Check second $\{\cdot, \cdot\}$ bracket is also zero:

$$\omega_\lambda([X^\#, Y^\#], Z^\#) = \omega_\lambda([X, Y]^\#, Z^\#) = \big\langle \lambda, \big[[X, Y], Z\big]\big\rangle.$$

So the second bracket becomes

$$\langle \lambda, -[[X, Y], Z] - [[Z, X], Y] - [[Y, Z], X]\rangle = 0$$

which follows from the Jacobi identity.

□

Proof: (of (3)): To show ω_λ is nondegenerate, for all $X \in \mathbf{g}$ (or $[X, \lambda] \in T_\lambda \mathcal{O}_\lambda$) we need to exhibit $Y \in \mathbf{g}$ with $\omega_\lambda([X, \lambda], [Y, \lambda]) \neq 0$. In fact by (3)

$$\$\omega_\lambda([X, \lambda], [Y, \lambda]) = \langle \lambda, [X, Y]\rangle$$

$$= \langle [\lambda, X], Y\rangle.$$

This is nonzero if we choose $Y = [\lambda, X]$. Then $\langle [\lambda, X], Y\rangle \neq 0$. □

Proof: (of (4)): The above gives

$$\Phi_X(Z) = \langle X, Z\rangle$$

for

$$\Phi_X : \mathcal{O}_\lambda \to \mathbf{g}^*$$

or

$$\Phi_X(\zeta) = \zeta(X)$$

for

$$\Phi_X : \mathcal{O}_\zeta \to \mathbf{g}^*.$$

in other words $\Phi : \mathcal{O}_\zeta \subset \mathbf{g}^* \to \mathbf{g}^*$ is the inclusion map. □