8 Equivariant Cohomology

References: Audin §5, Berline-Getzler-Vergne §7 .

Definition 8.1 *Let G be a compact Lie group. The* universal bundle *EG is a contractible space on which G acts freely.*

Definition 8.2 *The* classifying space *BG is $BG = EG/G$.*

Proposition 8.3

$$H^*(BG) = S(\mathbf{g}^*)^G = S(\mathbf{t}^*)^W$$

(polynomials on $\mathbf{g}$ invariant under the adjoint action, or polynomials on $\mathbf{t}$ invariant under the Weyl group action)

Here, the degree in $H^*(BG)$ is twice the degree as a polynomial on $\mathbf{g}$.
(Chern-Weil: evaluate polynomials on curvature)

Example 8.4 *S^1 acts freely on all S^{2n+1}, and these have homology only in dimensions 0 and $2n+1$. The universal space $EU(1)$ is*

$$S^\infty = \{(z_1, z_2, \ldots) \in \mathbb{C} \otimes \mathbb{Z} : \text{ only finitely many nonzero terms}, \sum_j |z_j|^2 = 1\}$$

$$= S^1 \cup S^3 \cup \ldots$$

where $S^{2n-1} \to S^{2n+1}$ via $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, 0)$. The space S^∞ is in fact contractible, so it is $EU(1)$.

Lemma 8.5

$$BU(1) = EU(1)/U(1) = \mathbb{C}P^\infty$$

Proposition 8.6 *$H^*(BU(1)) = \mathbb{C}[x]$ where x has degree 2.*

Recall

$$H^*(\mathbb{C}P^n) = \frac{\mathbb{C}[x]}{< x^{n+1} = 0 >}.$$

8.1 Homotopy quotients

Suppose M is a manifold acted on by a compact Lie group (not necessarily freely). We want to find a substitute for the cohomology of M/G, which is in general not a manifold.

Definition 8.7

$$H^*_G(M) = H^*(M_G)$$

where we define the homotopy quotient

$$M_G = (M \times EG)/G.$$

Proposition 8.8 *If G acts freely on M then M/G is a smooth manifold and*

$$H^*_G(M) = H^*(M/G)$$

More generally, $H^*_G(M)$ is a module over the ring

$$H^*_G = H^*_G(\mathrm{pt}) = H^*(BG).$$

8.2 The Cartan model

De Rham cohomology version of $H^*(M_G)$:

Definition 8.9

$$\Omega^*_G(M) = (\Omega^*(M) \otimes S(\mathbf{g}^*))^G$$

where we have defined

$$S(\mathbf{g}^*) = \{f : \mathbf{g} \to \mathbb{R} : f \textit{ is a polynomial}\}$$

Here, $S(\mathbf{g}^*)$ is acted on by G through the coadjoint action of G on $\mathbf{g}^*$.

Example 8.10 *T abelian:*

$$\Omega^*_T(M) = \Omega^*(M)^T \otimes S(\mathbf{t}^*)$$

since all polynomials on $\mathbf{t}$ are automatically invariant, because the action of T on $\mathbf{t}$ is trivial.

Lemma 8.11

$$S(\mathbf{t}^*) = \mathbb{C}[x_1, \dots, x_\ell]$$

where $\ell = \dim(T)$

Proposition 8.12

$$S(\mathbf{g}^*)^G = S(\mathbf{t}^*)^W$$

where the Weyl group W acts on $\mathbf{t}$

Let X be a vector field on M. An element $f \in \Omega^*_G(M)$ may be thought of as a G-equivariant map $f : \mathbf{g} \to \Omega^*(M)$, where the dependence of $f(X) \in \Omega^*(M)$ on $X \in \mathbf{g}$ is polynomial. The grading on $\Omega^*_G(M)$ is defined by $\deg(f) = n + 2p$ if $X \mapsto f(X)$ is p-linear in X and $f(X) \in \Omega^n(M)$. We may define a differential

$$D : \Omega^*_G(M) \to \Omega^*_G(M)$$

by

$$(Df)(X) = d(f(X)) - \iota_{X^\#} f(X)$$

where $X^\#$ is the vector field on M generated by the action of $X \in \mathbf{g}$ and ι denotes the interior product. Then $D \circ D = 0$ and D increases the degree in $\Omega^*_G(M)$ by 1.

Theorem 8.13 (Cartan [13]) *$H_G^*(M)$ is naturally isomorphic to the cohomology $H^*(\Omega_G^*(M), D)$ of this complex.*

In particular, since $D = 0$ on Ω_G^*, we see that $H_G^* = H_G^*(pt) = S(\mathbf{g}^*)^G$.

If (M, ω) is a symplectic manifold equipped with the Hamiltonian action of a compact group G, with moment map Φ, we define

$$\bar{\omega}(X) = \omega + \Phi_X \in \Omega_G^2(M).$$

Lemma 8.14

$$D\bar{\omega} = 0$$

Proof:

$$(D\bar{\omega})(X) = d\omega - i_{X^\#}\omega + d\Phi_X$$

But

$$i_{X^\#}\omega = d\Phi_X$$

by definition of Hamiltonian group actions. □

So $D\bar{\omega} = 0$ and we can define $[\bar{\omega}] \in H_G^2(M)$.

Definition 8.15 *A principal circle bundle over a point p equipped with the action of a torus T is $P = S^1$ equipped with a weight $\beta \in \mathrm{Hom}(T, U(1))$. Thus T acts on S^1 by*

$$t \in T : z \in S^1 \mapsto \beta(t)z.$$

Denote this bundle by P_β.

Example 8.16

$$T = U(1), \qquad \beta(t) = t^m$$

for $m \in \mathbb{Z}$

Denote this bundle by P_m.

8.3 Characteristic classes of bundles over $BU(1)$ and BT

Example 8.17

$$EU(1) \times_{U(1)} P_m$$

is the homotopy quotient of P_m. Explicitly this means

$$\{(z, w) \in EU(1) \times_{U(1)} P_m\}/\sim$$

where

$$(z, w) \sim (zu^{-1}, u^m w).$$

Every point (z, w) is equivalent to a point $(z', 1)$ by choosing $u = w^{-1/m}$ so

$$(z, w) \sim (zw^{1/m}, 1).$$

Since there are m solutions to $u = w^{-1/m}$, any two of which differ by multiplication by a power of $e^{2\pi i/m}$, we see that

$$EU(1) \times_{U(1)} P_m = EU(1)/\mathbb{Z}_m$$

where

$$\mathbb{Z}_m = \{e^{2\pi i r/m}, r = 0, \ldots, m-1\}.$$

Connection form θ_m on $EU(1) \times_{U(1)} P_m$:

Note that a connection form θ on $EU(1)$ satisfies $\int_{\pi^{-1}(b)} \theta = 1$. We also expect θ_m to satisfy

$$\int_{\pi_m^{-1}(b)} \theta_m = 1.$$

But since each fibre of $EU(1)$ may be written as $\{e^{i\phi}, \phi \in [0, 2\pi]\}$ and the fibre of $EU(1)/\mathbb{Z}_m \stackrel{\pi_m}{\to} BU(1)$ corresponds to $\{e^{i\phi}, \phi \in [0, 2\pi/m]\}$, we have $\int_{\pi_m^{-1}(b)} \theta = \frac{1}{m}$ so we need

Lemma 8.18

$$\theta_m = m\theta$$

in terms of our earlier θ.

It follows that the first Chern class $c_1(P_m)$ of the principal circle bundle P_m (which is represented in Chern-Weil theory by the curvature of $d\theta_m$) satisfies

Lemma 8.19

$$c_1(P_m)(X) = mX$$

where $c = c_1(EU(1) \to BU(1))$ is the generator of $H^*(BU(1))$.

This is a special case of

Lemma 8.20 *Let $T \to P \to M$ be a principal T-bundle with connection*

$$\theta = (\theta_1, \ldots, \theta_\ell) \in \Omega^1(P) \otimes \mathbf{t}.$$

Let $\beta \in \mathrm{Hom}(T, U(1))$. Form the associated principal circle bundle $P \times_T S^1 :=$

$$\{(p, s) \in P \times S^1\}/\sim$$

where $(p, s) \sim (pt, \beta(t^{-1})s)$ for $t \in T$. Write $\mathrm{Lie}(\beta) : \mathbf{t} \to \mathbb{R}$.

$$\mathrm{Lie}(\beta) = \{(b_1, \ldots, b_\ell)\}$$

($b_j \in \mathbb{Z}$). Then a connection form on $P \times_T S^1$ is

$$\sum_{j=1}^{\ell} b_j \theta_j = \mathrm{Lie}(\beta)(\theta).$$

Lemma 8.21 *If* $(x_1, \ldots, x_\ell) \in H^2(BT)$ *are the generators of* $H^*(BT)$ *for a torus* T *of rank* ℓ*, then the first Chern class of the associated principal circle bundle*

$$ET \times_T (S^1)_\beta$$

specified by the weight β *is*

$$c_1(ET \times_T (S^1)_\beta) = \sum_{j=1}^{\ell} b_j x_j.$$

8.4 Characteristic classes in terms of the Cartan model

Definition 8.22 *A* G*-equivariant vector bundle over a* G*-manifold* M *is a vector bundle* $V \to M$ *with an action of* G *on the total space covering the action of* G *on* M*.*

An equivariant principal circle bundle $P \to M$ *is a principal circle bundle with the action of* G *on the total space* P *covering the action of* M*.*

Lemma 8.23 *Suppose* $P \to M$ *is a principal circle bundle with connection* $\theta \in \Omega^1(P)$*. Its first Chern class is represented in de Rham cohomology by* $c_1(P) = [d\theta]$*.*

Lemma 8.24 *If* $P \xrightarrow{\pi} M$ *is a* G*-equivariant principal* $U(1)$*-bundle then its equivariant first Chern class is represented in the Cartan model by*

$$c_1^G(P) = [D\theta] = [d\theta - i_{X^\#}\theta]$$

where $X^\#$ *is the vector field on* P *generated by* $X \in \mathbf{g}$*.*

Proof: For a collection of sections $s_\alpha : U_\alpha \subset M \to P$, $s_\alpha^* D\theta$ is closed but not exact in $\Omega_G^*(M)$. In particular, if M is a point and $P = P_\beta = U(1)$ equipped with $\beta \in \mathrm{Hom}(T, U(1))$, then

$$c_1(P_\beta) = [D\theta] = -(\mathrm{Lie}(\beta))(X) = -i_{X^\#}\theta$$

(since $d\theta = 0$ on M). □

Lemma 8.25 *If* $P \to M$ *is a principal* $U(1)$ *bundle with* T *action, and the* T *action on* M *is trivial (but the* T *action on the total space of* P *is not trivial), then on each fibre* $\pi^{-1}(m) \cong S^1$*, the* T *action is given by a weight* $\beta \in \mathrm{Hom}(T, U(1))$*. Then*

$$c_1^T(P) = [D\theta] = [d\theta - \beta(X)].$$

Remark 8.26 *Atiyah-Bott p. 9 have a different convention on characteristic numbers. One obtains their convention from ours by replacing* X *by* $-X$*. Our convention is consistent with Berline-Getzler-Vergne §7.1.*

The situation of the preceding Lemma arises in the following context:

Lemma 8.27 *Let* M *be equipped with a* T *action, and let* F *be a component of* M^T*. For* $\alpha \in H_T^*(M)$ *and* $i_F : F \to M$ *the inclusion map,*

$$i_F^*\alpha \in H_T^*(F) = H^*(F) \otimes H_T^*(\mathrm{pt}) = H^*(F) \otimes \mathbb{R}[x_1, \ldots, x_\ell].$$

8.5 Equivariant first Chern class of a prequantum line bundle

Definition 8.28 *Let (M, ω) be a symplectic manifold with a Hamiltonian action of a group G. A* prequantum bundle with connection *is a principal circle bundle $P \to M$ for which $c_1(P) = [\omega]$, equipped with a connection θ for which $d\theta = \pi^*\omega$.*

Lemma 8.29 *If we impose the condition that $\mathcal{L}_{X^\#}\theta = 0$, then*

$$di_{X^\#}\theta = -i_{X^\#}d\theta$$

$$= -i_{X^\#}\omega = -d\Phi_X.$$

It is thus natural to also impose the condition

$$i_{X^\#}\theta = -\Phi_X.$$

Thus the specification of a moment map for the group action is equivalent to specifying a lift of the action of T from M to the total space P.

Lemma 8.30 *If $F \subset M$ is a component of the fixed point set, then*

$$i_F^*\bar{\omega}(X) \in \Omega_T^2(F)$$

$$= \omega|_F + \Phi_X(F).$$

Proof: For any $f \in F$, $P|_f$ is a copy of S^1 on which T acts using a weight $\beta_F \in \mathrm{Hom}(T, U(1))$.

$$\mathrm{Lie}(\beta_F) \in \mathrm{Hom}(\mathbf{t}, \mathbb{R}).$$

The equivariant first Chern class of P is

$$c_1^T(P)|_F = c_1(P)|_F - \mathrm{Lie}(\beta_F) = [\omega]|_F - \mathrm{Lie}(\beta_F).$$

Identifying the two equivariant extensions of $\omega|_F$ we see that

$$\Phi_X(F) = -\mathrm{Lie}(\beta_F)(X).$$

At fixed points of the action, the value of the moment map is a weight: provided the symplectic form ω satisfies $[\omega] = c_1(P)$ for some principal S^1-bundle P. This is true iff $[\omega] \in H^2(M, \mathbb{Z})$. □

8.6 Euler classes and equivariant Euler classes

References: Roe, Gilkey, Milnor-Stasheff Appendix C

Definition 8.31 *If E is a complex vector bundle of rank m (write this as $E_{\mathbb{C}}$) then we may regard it as a real vector bundle of rank $2m$ (write this as $E_{\mathbb{R}}$).*

Definition 8.32 *The Euler class of E is a characteristic class $e(E)$ associated with real vector bundles $E \to M$ of rank r, if r is the (real) dimension of M.*

Definition 8.33 *If $E_{\mathbb{C}}$ is a complex vector bundle of (complex) rank m, then $e(E_{\mathbb{R}}) = c_m(E_{\mathbb{C}})$.*

Proposition 8.34 *(Euler class is multiplicative) If $E = E_1 \oplus E_2$, then $e(E) = e(E_1)e(E_2)$.*

Proposition 8.35 *If $E = L_1 \oplus \ldots \oplus L_m$ (direct sum of line bundles) then $e(E) = c_1(L_1) \ldots c_1(L_m)$.*

Proposition 8.36 *If E is a complex vector bundle with a T action, and $E = \sum_j L_j$ where the L_j are complex line bundles with T action given by weights $\beta_j : T \to U(1)$, then the equivariant Euler clas of E is*

$$e^T(E) = \prod_j c_1^T(L_j)$$

which is represented in the Cartan model by

$$e^T(E)(X) = \prod_j (d\theta_j - (\mathrm{Lie}\beta_j)(X)).$$

We can usually reduce to this situation by the *splitting principle*: see Bott-Tu §21.

Example 8.37 *If T acts on M and F is a component of M^T, then the normal bundle ν_F is a T-equivariant bundle over F (T acts trivially on F, but not on ν_F). Assume ν_F decomposes equivariantly as $\sum_j \nu_{F,j}$ with weights $\beta_{F,j}$.*

The equivariant Euler class $e_F := e^T(\nu_F)$ is then given by

$$e_F(X) = \prod_j (c_1(\nu_{F,j}) - \beta_{F,j}(X)).$$

(References: Berline-Getzler-Vergne §7.2; Audin Chap. V.6).

8.7 Localization formula for torus actions

If M is a G-manifold of dimension m, then the equivariant pushforward is

$$\int_M : H_G^*(M) \to H_G^*(\mathrm{pt}).$$

Topologically this is the pairing with the fundamental class of M. In the Cartan model, we represent α by $\eta \in \Omega_G^*(M)$ satisfying $D\eta = 0$. Stokes' theorem implies that if M has no boundary then $\int_M D\alpha(X) = 0$ for any $\alpha \in \Omega^*_G(M)$. Thus $\int_M \eta(X)$ depends only on the class of η in the cohomology of the Cartan model. Define

$$Z^*_G(M) = \{\alpha \in \Omega^*_G(M) : D\alpha = 0\}$$

and

$$B^*_G(M) = D\Omega^*_G(M).$$

So

$$H^*_G(M) = Z^*_G(M)/B^*_G(M)$$

and

$$\int_M : H^*_G(M) \to H^*_G(\mathrm{pt}).$$

Remark 8.38 *The integral*

is a smooth function of X (cf. $\sin X/X$ is a smooth function of X). But the terms corresponding to individual F are meromorphic functions of X which do have poles. These poles cancel in the sum over F.

8.8 Proof of localization theorem when M^T consists of isolated fixed points

In this case $e_F(X) = (-1)^n \prod_j \beta_{F,j}(X)$. This implies the dimension of M is even, since nontrivial irreducible representations of T have real dimension 2, and if F is a fixed point, $T_F M$ must decompose as a direct sum of nontrivial irreducible representations of T. (If there were any subspaces of $T_F M$ on which T acted trivially, they would be tangent to the fixed point set M^T, but we have already assumed M^T consists of isolated fixed points.

Lemma 8.39 *Let θ be any 1-form on M for which $\theta(X^\#) = 0$ iff ${x^\#}_m = 0$. Then on $M \setminus M^T$ we have that if $\alpha \in \Omega^*_T(M)$ and $D\alpha = 0$,*

$$\alpha = D\left(\frac{\theta\alpha}{D\theta}\right).$$

Proof:

(a) D is an antiderivation (because d and $i_{X^\#}$ are antiderivations)

(b) So $D(\theta\alpha) = (D\theta)\alpha$ (since $D\alpha = 0$ and

$$\alpha = D\left(\frac{\theta\alpha}{D\theta}\right)$$

(we use that $D(f/D\theta) = Df/D\theta$).

(c) The formal expression $\frac{\theta\alpha}{D\theta}$ makes sense on $M \setminus M^T$ since $D\theta = d\theta - \theta(X^\#)$ and $\theta(X^\#) \neq 0$ on $M \setminus M^T$. Then

$$\frac{1}{D\theta} = \frac{1}{-\theta(X^\#}\left(1 - \frac{d\theta}{\theta(X^\#)}\right)$$

$$= \frac{-1}{\theta(X^\#)}\sum_{r\geq 0}\left(\frac{d\theta}{\theta(X^\#)}\right)^r$$

and $(d\theta)^r = 0$ for $2r > \dim(M)$. So the series only has a finite number of nonzero terms. □

Lemma 8.40 *There exist θ satisfying the hypotheses of the previous lemma (cf. inverse of equivariant Euler class). We may construct θ on M as follows. Denote it by θ'. Choose a T-invariant metric g on M and define for $\xi \in T_m M$*

$$\theta'_m({X^\#}_m,) = g({X^\#}_m, {X^\#}_m)$$

Then $\theta'_m({X^\#}_m) = g({X^\#}_m, {X^\#}_m) = 1$ on $M \setminus M^T$. In a neighbourhood of $F \in M^T$, we shall take a differnet choice of θ: denote it by θ^H. Choose coordinates $(x_1, \ldots, x_{2n-1}, x_{2n})$ on $T_F M \cong \mathbb{C} \oplus \ldots \mathbb{C}$ (n copies of $\mathbb{C}$) for which T acts on j-th copy of $\mathbb{C}$ (with coordinates $z_j = x_{2j-1} + ix_{2j}$) by linear action with weight $\beta_j \in$ $\mathrm{Hom}(T, U(1))$, $\mathrm{Lie}(\beta_j) : \mathbf{t} \to \mathbb{R}$.

Define $\mathrm{Lie}(\beta_j)(X) = \lambda_j \in \mathbb{R}$ *for a specific $X \in \mathbf{t}$ for which all the* $\mathrm{Lie}(\beta_j)(X)$ *are nonzero. (This is true for almost all $X \in \mathbf{t}$.)*

On $\mathbb{C}^n \cong T_F M$, define

$$\theta' = \sum_j \frac{1}{\lambda_j}(x_{2j-1}dx_{2j} - x_{2j}dx_{2j-1}).$$

Using the exponential map as defined in differential geometry:

$$\exp : T_F M \to M$$

This map is T-equivariant.

So

$$(x_1, \ldots, x_{2n})$$

become coordinates on an open neighbourhood U_F of F in M, and in these coordinates, the action of T is still given by the linear action on $\mathbb{C}^n$ for which the action on the j-th copy of $\mathbb{C}$ is given by the weight β_j.

Using a partition of unity, construct a smooth T-invariant function

$$f : M \to [0, 1]$$

with $f = 0$ on $M \setminus U_F$. Choose an open neighbourhood $F \in V_F \subset U_F$ (for instance U_F is a ball of radius 2, and V_F is a ball of radius 1) and require $f = 1$ on V_F. Then define

$$\theta = (1-f)\theta' + f\theta''$$

Thus

$$\theta|_{M \setminus \cup_{F \in M^T} U_F} = \theta'$$

and

$$\theta|_{V_F} = \theta''$$

and for appropriately chosen f, $\theta_m(X^\#) \neq 0$ when $m \notin M^T$.

Lemma 8.41 *(Stokes' Theorem for Cartan model for manifolds with boundary) Let M be a manifold with boundary ∂M, with G action such that the action of G sends ∂M to ∂M. If $\alpha \in \Omega_G^*(M)$ then*

$$\int_M D\alpha = \int_{\partial M} \alpha.$$

Proof: Decompose $\alpha = \alpha_0 + \ldots + \alpha_{\dim M}$ where α_j is a differential form of degree j (depending on X). Then $\int_M \alpha := \int_M \alpha_{\dim M}$ (the other α_j contribute 0, by definition). Then $(D\alpha)_{\dim M} = d\alpha_{\dim M - 1}$ (since the $i_{X^\#}$ part of the Cartan model differential reduces degree of forms so it cannot contribute). Now apply ordinary Stokes' Theorem to $(D\alpha)_{\dim M}$. $\square$

Let $B_\epsilon(F) \subset \exp(U_F)$ be a ball of radius ϵ around F (in the local coordinates on $\exp(U_F)$). Then

$$\begin{aligned}\int_M \alpha &= \lim_{\epsilon \to 0} \int_{M \setminus \cup_F B_\epsilon(F)} \alpha \\ &= \lim_{\epsilon \to 0} \int_{M \setminus \cup_F B_\epsilon(F)} D\left(\frac{\theta\alpha}{D\theta}\right) \\ &= -\lim_{\epsilon \to 0} \sum_F \int_{\partial B_\epsilon(F)} \frac{\theta\alpha}{D\theta}\end{aligned}$$

(by Stokes). Define $\partial B_\epsilon(F) = S_\epsilon(F)$, a sphere of radius ϵ in $\mathbb{C}^n$.

$$S_\epsilon(F) = \{(x_1, \ldots, x_{2n}) : \sum_j |x_j|^2 = \epsilon^2\}.$$

Define $\phi : S^{2n-1} \to S_\epsilon(F)$ by $\phi(\bar{x}) = \epsilon\bar{x}$.

8.9 Equivariant characteristic classes

Define the pushforward map $\pi_* : \Omega^*_G(M) \to \Omega^*_G$ by $\pi_*(\eta)(X) = \int_M \eta(X)$.

Stokes' Theorem for equivariant cohomology in the Cartan model tells us that if M is a G-manifold with boundary and $G : \partial M \to \partial M$ (where the action of G on ∂M is locally free) and $\eta \in \Omega^*_G(M)$, then

$$\int_M (D\eta)(X) = \int_{\partial M} \eta(X).$$

It follows that the pushforward map π_* induces a map $H^*_G(M) \to H^*_G$.

Definition 8.42 *Suppose E is a (complex) vector bundle on a manifold M equipped with a Hamiltonian action of a group G which lifts the action of G on M. The equivariant Chern classes $c_r^G(E)$ are given by*

$$c_r^G(E) = c_r(E \times_G EG \to M \times_G EG).$$

Likewise the equivariant Euler class *of E is given by*

$$e^G(E) = e(E \times_G EG \to M \times_G EG).$$

Example 8.43 Equivariant characteristic classes in the Cartan model. *Suppose E is a complex vector bundle of rank N on a manifold M equipped with the action of a group G. Let ∇ be a connection on E compatible with the action of G. Define the* moment *of E, $\tilde{\mu} \in G(\mathrm{End}E \otimes \mathbf{g}^*)$ (see [6], Section 7.1) as follows:*

$$\mathcal{L}_{X^\#} s - \nabla_{X^\#} s = \tilde{\mu}(X) s \tag{1}$$

for $s \in G(E)$ (where $X \in G$ and $X^\#$ is the vector field on M defined by the action of G). Notice that the action of G on the total space of E permits us to define the Lie derivative $\mathcal{L}_{X^\#} s$ of a section $s \in G(E)$, and that the formula (1) defines $\tilde{\mu}$ as a zeroth order operator (i.e. a section of End$E$ *depending linearly on $X \in \mathbf{g}$).*

We find that the representatives in the Cartan model of $c_r^G(E)$ are given by

$$c_r^G(E) = [\tau_r(F_\nabla + \tilde{\mu}(X))]$$

where $F_\nabla \in G(\mathrm{End}E \otimes \Omega^2(M))$ is the curvature of ∇ and τ_r is the elementary symmetric polynomial of degree r on $\mathbf{u}(N)$ giving rise to the Chern class c_r.

Remark 8.44 *If M is symplectic and E is a complex line bundle $\mathcal{L}$ whose first Chern class is the De Rham cohomology class of the symplectic form, then the moment defined in Example 8.43 reduces to the symplectic moment map for the action of G.*

Example 8.45 *Suppose E is a complex line bundle over M equipped with an action of a torus T compatible with the action of T on M, and denote by F the components of the fixed point set of T over M. Suppose a torus T acts on the fibres of $E|_F$ with weight $\beta_F \in \mathbf{t}^*$: in other words* $\exp(X) \in T : z \in E|_F \mapsto e^{i\beta_F(X)}z$. *Then*

$$e^T(E)|_F = c_1(E) + \beta_F(X).$$

Example 8.46 *If G acts on a manifold M, bundles associated to M (e.g. tangent and cotangent bundles) naturally acquire a compatible action of G.*

Example 8.47 *Suppose a torus T acts on M and let F be a component of the fixed point set. (Notice that each F is a manifold, since the action of T on the tangent space T_fM at any $f \in F$ can be linearized and the linearization gives charts for F as a manifold.) Let ν_F be the normal bundle to F in M; then T acts on ν_F. Without loss of generality (using the splitting principle: see for instance [10]) we may assume that ν_F decomposes T-equivariantly as a sum of line bundles $\nu_{F,j}$ on each of which T acts with weight $\beta_{F,j} \in \mathbf{t}^*$. Thus one observes that the equivariant Euler class of ν_F is*

$$e_F(X) = \prod_j (c_1(\nu_{F,j}) + \beta_{F,j}(X)).$$

Notice that $\beta_{F,j} \neq 0$ for any j (since otherwise $\nu_{F,j}$ would be tangent to the fixed point set rather than normal to it). We may thus define

$$e_F^0(X) = \prod_j \beta_{F,j}(X)$$

and we have

$$e_F(X) = e_F^0(X) \prod_j (1 + c_1(\nu_{F,j})/\beta_{F,j}(X)).$$

Since $c_1(\nu_{F,j})/\beta_{F,j}(X)$ is *nilpotent*, we find that we may define the inverse of $e_F(X)$ by

$$\frac{1}{e_F(X)} = \frac{1}{e_F^0(X)} \sum_{r=0}^{\infty} (-1)^r (c_1(\nu_{F,j})/\beta_{F,j}(X))^r;$$

only a finite number of terms contribute to this sum.

Example 8.48 $U(1)$ **actions with isolated fixed points.**

Suppose the action of $T \in U(1)$ on M has isolated fixed points. Suppose the normal bundle $\nu_F = T_FM$ at each fixed point F decomposes as a direct sum $\nu_F \cong \oplus_{j=1}^N \nu_{F,j}$ where each $\nu_{F,j} \cong \mathbb{C}$ and M acts with multiplicity $\mu_{F,j}$ on $\nu_{F,j}$ (for $0 \neq \mu_{F,j} \in \mathbb{Z}$): in other words

$$t \in U(1) : z_j \in \nu_{F,j} \mapsto t^{\mu_{F,j}} z_j.$$

We then find that the equivariant Euler class is

$$e_F(X) = (\prod_j \mu_{F,j}) X^N.$$

8.10 The abelian localization theorem

A very important localization formula for equivariant cohomology with respect to torus actions is given by the following theorem.

Theorem 8.49 (Berline-Vergne [8]; Atiyah-Bott [3]) *Let T be a torus acting on a manifold M, and let $\mathcal{F}$ index the components F of the fixed point set M^T of the action of T on M. Let $\eta \in H_T^*(M)$. Then*

$$\int_M \eta(X) = \sum_{F\in\mathcal{F}} \int_F \frac{\eta(X)}{e_F(X)}.$$

Proof 1: **(Berline-Vergne [8])** Let us assume $T = U(1)$ for simplicity. Define $M_\epsilon = M - \coprod_F U_\epsilon^F$ where U_ϵ^F is an ϵ-neighbourhood (in a suitable equivariant metric) of the component F of the fixed point set M^T. On M_ϵ, T acts locally freely, so we may choose a connection θ on M_ϵ viewed as the total space of a principal (orbifold) $U(1)$ bundle (in other words, θ is a 1-form on M_ϵ for which $\theta(V) = 1$ where V is the vector field generating the S^1 action). Now for every equivariant form $\eta \in \Omega_T^*(M)$ for which $D\eta = 0$, we have that

$$\eta = D\Big(\frac{\theta\eta}{d\theta - X}\Big).$$

Applying the equivariant version of Stokes' theorem we see that

$$\int_M \eta(X) = \lim_{\epsilon\to 0}\int_{M_\epsilon} \eta(X) = \sum_F \lim_{\epsilon\to 0}\int_{\partial U_\epsilon^F} \frac{\theta\eta(X)}{d\theta - X}.$$

It can be shown (see [8] or Section 7.2 of [6]) that as $\epsilon \to 0$, $\int_{\partial U_\epsilon^F} \frac{\theta\eta(X)}{d\theta - X}$ tends to $\int_F \frac{\eta(X)}{e_F(X)}$. □

Proof 2: (**Atiyah-Bott[3]**) We work with the functorial properties of the pushforward (in equivariant cohomology) under the map i_F including F in M. We see that $i_F^*(i_F)_* = e_F$ is multiplication by the equivariant Euler class e_F of the normal bundle to F. Further one may show ([30], Section 6, Proposition 8) that the map

$$\sum_F i_F^* : H_T^*(M) \mapsto \oplus_{F\in\mathcal{F}} H^*(F) \otimes H_T^*$$

is *injective*. Thus we see that each class $\eta \in H_T^*(M)$ satisfies

$$\eta = \sum_{F\in\mathcal{F}} (i_F)_* \frac{1}{e_F} i_F^* \eta \tag{2}$$

(by applying i_F^* to both sides of the equation). Now $\int_M \eta = \pi_*\eta$ (where the map $\pi : M \to \mathrm{pt}$ and $\pi_* : H_T^*(M) \to H_T^*$ is the pushforward in equivariant cohomology). The result now follows by applying π_* to both sides of (2) (since $\pi_* \circ (i_F)_* = (\pi_F)_* = \int_F$). $\square$

Proof 3: **(Bismut [9]; Witten [40], 2.2.2)** Let $\lambda \in \Omega^1(M)$ be such that $\iota_{X^\#}\lambda = 0$ if and only if $X^\# = 0$: for instance we may choose $\lambda(Y) = g(X^\#, Y)$ for any tangent vector Y (where g is any G-invariant metric on M). We observe that if $D\eta = 0$ then $\int_M \eta(X) = \int_M \eta(X) e^{tD\lambda}$ for any $t \in \mathbb{R}$. Now $D\lambda = d\lambda - g(X^\#, X^\#)$, so

$$\int_M \eta(X) e^{tD\lambda} = \int_M \eta(X) e^{-tg(X^\#, X^\#)} \sum_{n \geq 0} t^n (d\lambda)^n / n!.$$

Taking the limit as $t \to \infty$ we see that the integral reduces to contributions from points where $X^\# = 0$ (i.e. from the components F of the fixed point set of T). A careful computation yields Theorem 8.49.[2]

[2]The technique used in this proof – introducing a parameter t, showing independence of t by a cohomological argument and showing localization as t tends to some limit – is by now universal in geometry and physics. Two of the original examples were Witten's treatment of Morse theory in [38] and the heat equation proof of the index theorem [4].