MAT1312F Exercises 3

Due date: November 30, 2016

(1) **(Equivariant cohomology of subgroups)**

(a) Let G be a compact Lie group. Suppose $H \subset G$ is a Lie subgroup of G. Show that there is a map

$$H^*_G(M) \to H^*_H(M).$$

(Either use the Cartan model, or else use the existence of a fibration $M \times_H EH \to M \times_G EG$. This fibration exists since, as EG is a contractible space on which G acts freely, it is also a contractible space on which H acts freely, and hence may be identified with EH.)

(b) Show that in particular there is a map $H^*_G(M) \to H^*_T(M)$ where T is the maximal torus of G.

(c) In (b), let M be a point. Using the Cartan model to identify $H^*_G(pt) \cong S(\mathfrak{g}^*)^G$ and $H^*_T(pt) \cong S(\mathfrak{t}^*)$, show that the image of the map $H^*_G(pt) \to H^*_T(pt)$ (which is in fact isomorphic to $H^*_G(pt)$) is $S(\mathfrak{t}^*)^W \subset S(\mathfrak{t}^*)$.

(d) Show that if $G = SU(2)$ and $T = U(1)$ the image of $H^*_G(pt)$ in $H^*_T(pt) \cong \mathbb{R}[X]$ is the polynomials of even degree in X.

(2) **(Functorial properties of equivariant cohomology)** Show that if M_1 and M_2 are two manifolds equipped with actions of G and $f : M_1 \to M_2$ is a G-equivariant map then it induces a map $f^* : H^*_G(M_2) \to H^*_G(M_1)$. (This correspondence has the property that if M_3 is another manifold with a G action and $g : M_2 \to M_3$ is G equivariant then $(g \circ f)^* = f^* \circ g^*$.)

(3) (a) Show that if G is a compact Lie group and H is a Lie subgroup of G then

$$H^*_G(G/H) \cong H^*_H(pt).$$

(Hint: rewrite $EG \times_G G/H$ as a quotient by H.)

(b) Let $G = SO(3)$ and $H = U(1)$. What is the ring $H^*_G(S^2)$ where $SO(3)$ acts on S^2 by rotation? (Hint: write $S^2 = SO(3)/U(1)$.)

(4) **(Functorial properties of pushforward)** Let M be a G-manifold. Show that if $\alpha \in S(\mathfrak{g}^*)^G$ and $\pi^*\alpha$ is its inclusion as an element of $\Omega^*_G(M)$, and $\beta \in \Omega^*_G(M)$, then

$$\int_M (\beta \pi^* \alpha) = (\int_M \beta)\alpha :$$

It follows that the map $\int_M : H^*_G(M) \to H^*_G(pt)$ is a homomorphism of $H^*_G(pt)$-modules.

(5) **(Duistermaat-Heckman for the two-sphere)**

Let $T = U(1)$ act on $M = S^2$ by rotation about the z axis, so that the moment map is $\mu_X(\phi, z) = Xz$.

(a) Compute the integral

$$\int_M \omega e^{i\mu_X}$$

by elementary methods.

(b) Compute the same integral using the Duistermaat-Heckman theorem.

(c) Show that the pushforward of the Liouville measure on S^2 under the moment map μ is the characteristic function $\chi_{[-1,1]}$ of the interval $[-1,1]$, defined by

$$\chi_{[-1,1]}(\xi) = 1 \text{ if and only if } |\xi| \leq 1.$$

(d) Show that the Fourier transform of $\chi_{[-1,1]}$ is

$$h(X) = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{iX}}{iX} + \frac{e^{-iX}}{-iX}\right).$$

(This is a constant multiple of the answer found in (a) and (b).)

(e) Show that if one wishes to define a value for the Fourier transform of

$$h(X) = \frac{e^{i\mu X}}{iX}$$

which is supported on $[b,\infty)$ for some $b \in \mathbb{R}$, one should define the Fourier transform to be

$$\hat{h}(\xi) = -\sqrt{2\pi}H(\xi - \mu),$$

where $H(\xi)$ is the Heaviside function

$$H(\xi) = 1, \xi > 0;$$

$$H(\xi) = 0, \xi \leq 0.$$

Hint: use the equation

$$\frac{d}{d\xi}H(\xi) = \delta(\xi).$$

(This is a special case of the construction of Guillemin-Lerman-Sternberg for Fourier transforms of the terms entering in the Duistermaat-Heckman formula for the oscillatory integral over M.)

(f) Using (e), recover the result that the Fourier transform of the function h found in (d) is

$$\chi_{[-1,1]}(\xi) = H(\xi+1) - H(\xi-1).$$

(This is a special case of Guillemin-Lerman-Sternberg's characterization of the pushforward of the Liouville measure under the moment map as a sum of piecewise polynomial functions supported on a collection of affine cones, the apex of each of which is $\mu(F)$ for some fixed point F of the torus action.)

(6) **(Harish - Chandra formula)** Prove that if G is a compact connected Lie group with maximal torus T, and $\mathcal{O}_\lambda$ is the orbit of the coadjoint action on $\mathfrak{g}^*$ through a point $\lambda \in \mathfrak{t}^*_+$, then if $X \in \mathfrak{t}$ we have the following formula for integrals of certain functions over the coadjoint orbit $\mathcal{O}_\lambda$:

$$\int_{\xi\in\mathcal{O}_\lambda} e^{<\xi,X>}\frac{\omega^N}{N!} = \sum_{w\in W}(-1)^w \frac{e^{<w\lambda,X>}}{\prod_{\gamma>0}\gamma(X)}.$$

Here, ξ denotes the integration variable in $\mathfrak{g}^*$, $< \cdot, \cdot >$ is the canonical pairing $\mathfrak{g}^* \otimes \mathfrak{g} \to \mathbb{R}$ and γ denote the positive roots. (This exercise generalizes part (b) of the previous one.)

(7) **(Quantization of S^2)** (a) Assume the symplectic form ω of S^2 has been normalized so that the prequantum line bundle $\mathcal{L}$ over S^2 is the hyperplane line bundle. If $k \geq 0$, show the space $\mathcal{H}^k$ of sections of $\mathcal{L}^k$ is the space of homogeneous polynomials of degree k in two variables, which has dimension $k+1$.

(b) What are the weights of the natural action of $U(1)$ on $\mathcal{H}^k$ (induced from the rotation action on $S^2 = \mathbb{C}P^1$ defined by by $u : [z_0 : z_1] \mapsto [uz_0 : u^{-1}z_1]$)?

(c) What is the moment polytope corresponding to the symplectic form $k\omega$?