MAT1312 Exercises 1

Due date: October 5, 2016

1. Let $f : \mathbb{C} \to \mathbb{R}$ be the moment map for the standard action of $U(1)$ on $\mathbb{C}$ by rotation:

$$f : z \mapsto -|z|^2/2$$

Construct the Hamiltonian flow of f^2 and f^3. Show that the orbits are all periodic but of different period depending on the value of $|z|$: find the period of the orbit as a function of the radius. (Thus the functions f^2 and f^3 are NOT moment maps for a circle action, although all orbits are periodic.)

2. (Coadjoint orbits in $u(n)$)

Recall that any matrix in $u(n)$ may be conjugated to a matrix of the form

$$\mathrm{diag}(i\lambda_1, \ldots, i\lambda_n),$$

where the $\lambda_j \in \mathbb{R}$ and $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.

Let $G = U(n)$ and let M be the orbit of $\mathrm{diag}(i\lambda_1, \ldots, i\lambda_n)$ in the Lie algebra $\mathfrak{g}$ of $U(n)$:

$$M = \{A \in GL(n, \mathbb{C}) : A + A^\dagger = 0, \quad \text{A has eigenvalues } i\lambda_1, \ldots, i\lambda_n\}.$$

Let T be the maximal torus of $U(n)$.

Show that the moment map for the action of T on M is the projection

$$A \to (A_{11}, \ldots, A_{nn})$$

onto the diagonal elements of the matrix.

3. Let T be a torus acting on a symplectic manifold M in a Hamiltonian way, and let $F \subset M^T$ be a component of the fixed point set. Show that $\mu(F)$ is a point. (Hint: show that for any $f \in F$, $d\mu_f = 0$.)

4. Show that the Hamiltonian vector fields of the components of the map

$$\mu : T^*\mathbb{R}^3 \to \mathbb{R}^3$$

given by the cross product,

$$\mu : (\bar{q}, \bar{p}) \mapsto \bar{q} \times \bar{p}$$

are the vector fields $\hat{X}$ on $T^*\mathbb{R}^3$ generated by the action of $X \in \mathbb{R}^3 = \mathfrak{g}$ on $T^*\mathbb{R}^3$ (where $G = SO(3)$ acts on $\mathbb{R}^3$ by rotations).

5. (a) Show that if a submanifold N of a symplectic manifold M is preserved by the action of an almost complex structure J on M (in other words N is an almost complex submanifold of M with respect to J) then the symplectic form restricts to a nondegenerate form on N.

(b) Assume that $f : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function. Show that if 0 is a regular value of f then $f^{-1}(0)$ is a symplectic submanifold of $\mathbb{C}^n$.

(c) Assume that $f : \mathbb{C}^n \to \mathbb{C}$ is a *homogeneous* polynomial function (in other words $f(\lambda z) = \lambda^d f(z)\ \forall \lambda \in \mathbb{C}^*$). Show that if 0 is a regular value of f then $\{[z_1 : \ldots : z_n] \in \mathbb{C}P^{n-1} : (z_1, \ldots, z_n) \in f^{-1}(0)\}$ is a symplectic submanifold of $\mathbb{C}P^{n-1}$.

6. (Orbits of Hamiltonian group actions are isotropic) Let M be equipped with the Hamiltonian action of a compact Lie group G. Show that the orbits of the action of G on $\mu^{-1}(0)$ are isotropic with respect to the symplectic structure.

7. (Symplectic slices) Let Y be a symplectic manifold equipped with the Hamiltonian action of a torus T which is the maximal torus of a compact Lie group G with moment map $\mu_T : Y \to \mathfrak{t}^*$.

   Define
   $$M := Y \times_T G = \{(y, g) \in Y \times G : (y, g) \simeq (ty, tg) \text{ for } t \in T\}.$$

   Define a symplectic structure on M on with respect to which the action of G is Hamiltonian. Exhibit a moment map $\mu_G : M \to \mathfrak{g}^*$ for the action of G on M. What is $\mu_G^{-1}(\mathfrak{t})$?

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8. Show explicitly that the diagonal elements of matrices conjugate (under $SU(2)$) to $\mathrm{diag}(2\pi i, -2\pi i)$ in $\mathfrak{su}(2)$ are of the form $\theta \mathrm{diag}(2\pi i, -2\pi i)$ where $\theta \in [-1, 1]$.