MAT1312F Exercises 3

Due date 11th week (week of March 26, 2012)

(1) **(Equivariant cohomology of subgroups)**

(a) Let G be a compact Lie group. Suppose $H \subset G$ is a Lie subgroup of G. Show that there is a map

$$H_G^*(M) \to H_H^*(M).$$

(Either use the Cartan model, or else use the existence of a fibration $M \times_H EH \to M \times_G EG$. This fibration exists since, as EG is a contractible space on which G acts freely, it is also a contractible space on which H acts freely, and hence may be identified with EH.)

Solution: Because H acts freely on EG, EG is homotopy equivalent to EH. so we have

$$f : EG \times_H M \to EG \times_F M.$$

This leads to $f^* : H_G^*(M) \to H_H^* M$.

(b) Show that in particular there is a map $H_G^*(M) \to H_T^*(M)$ where T is the maximal torus of G.

Solution: This is the special case $H = T$.

(c) In (b), let M be a point. Using the Cartan model to identify $H_G^*(pt) \cong S(\mathfrak{g}^*)^G$ and $H_T^*(pt) \cong S(\mathfrak{t}^*)$, show that the image of the map $H_G^*(pt) \to H_T^*(pt)$ (which is in fact isomorphic to $H_G^*(pt)$) is $S(\mathfrak{t}^*)^W \subset S(\mathfrak{t}^*)$.

Solution: $H_G^*(pt)$ is the set of G-invariant polynomials on $\mathfrak{g}$ (using the Cartan model, where the differential forms are constant functions). A G-invariant polynomial on $\mathfrak{g}$ is the same as a W-invariant polynomial on $\mathfrak{t}$.

Likewise, the inclusion of $H_G^*(pt) \to H_T^*(pt)$ is the inclusion map from the W-invariant polynomials on $\mathfrak{t}$ into the set of all polynomials on $\mathfrak{t}$.

(d) Show that if $G = SU(2)$ and $T = U(1)$ the image of $H_G^*(pt)$ in $H_T^*(pt) \cong \mathbb{R}[X]$ is the polynomials of even degree in X.

Solution: The Weyl group of $SU(2)$ is $\mathbb{Z}_2$, with the generator acting by multiplication by -1 on the generator of $\mathfrak{t}$. This means that the W-invariant polynomials on $\mathfrak{t}$ are the polynomials in one variable X of even degree in X.

(2) **(Functorial properties of equivariant cohomology)** Show that if M_1 and M_2 are two manifolds equipped with actions of G and $f : M_1 \to M_2$ is a G-equivariant map then it induces a map $f^* : H_G^*(M_2) \to H_G^*(M_1)$. (This correspondence has the property that if M_3 is another manifold with a G action and $g : M_2 \to M_3$ is G-equivariant then $(g \circ f)^* = f^* \circ g^*$.)

Solution:

Define $F : EG \times M_1 \to EG \times M_2$ by $F(x, m_1) = (x, f(m_1))$

This map is G-equivariant because

$$F(gx, gm_1) = (gx, f(gm_1) = (gx, gf(m_1)) = g(x, f(m_1) = gF(x, m_1).$$

This means the map descends to a map from $(EG \times M_1)/G$ to $(EG \times M_2)/G$.

(3) (a) Show that if G is a compact Lie group and H is a Lie subgroup of G then

$$H_G^*(G/H) \cong H_H^*(pt).$$

(Hint: rewrite $EG \times_G G/H$ as a quotient by H.)

Solution:

$$EG \times_G (G/H) \cong EG/H$$

but H acts freely on EG so EG is homotopy equivalent to EH so

$$EG/H \simeq EH/H \cong BH.$$

Hence

$$H^*(EG \times_G (G/H)) \cong H^*(BH) = H_H^*(\mathrm{pt}).$$

(b) Let $G = SO(3)$ and $H = U(1)$. What is the ring $H_G^*(S^2)$ where $SO(3)$ acts on S^2 by rotation? (Hint: write $S^2 = SO(3)/U(1)$.)

Solution: In this case

$$H_G^*(G/H) \cong H_H^*(\mathrm{pt})$$

(by part (a))

$$\cong \mathbb{Z}[X]$$

where X is a generator of degree 2.

(4) **(Functorial properties of pushforward)** Let M be a G-manifold. Show that if $\alpha \in S(\mathfrak{g}^*)^G$ and $\pi^*\alpha$ is its inclusion as an element of $\Omega_G^*(M)$, and $\beta \in \Omega_G^*(M)$, then

$$\int_M (\beta\pi^*\alpha) = (\int_M \beta)\alpha :$$

It follows that the map $\int_M : H_G^*(M) \to H_G^*(pt)$ is a homomorphism of $H_G^*(pt)$-modules.

Solution:

Using the Cartan model, $\alpha(X)$ is a polynomial in the variable $X \in \mathfrak{g}$ and it is constant as a differential form on M. So the integral of $\beta\alpha$ is equal to α times the integral of β, since α is constant when viewed as a differential form on M.