MAT1312 Exercises 1

Due date: January 30, 2020

(1) Show that the symplectic volume of a toric manifold is equal to the Euclidean volume of its moment polytope.

Solution: On a preimage of the interior of the moment polytope, there are defined action-angle coordinates x_i, θ_i where the symplectic form is defined as $\sum_i dx_i \wedge d\theta_i$ and the moment map sends $(x_1, \theta_1, \ldots, x_n, \theta_n) \mapsto (x_1, \ldots, x_n)$. It follows that the symplectic volume is $(2\pi)^n \int_B dx_1 \wedge \ldots dx_n$. where B is the moment polytope.

(2) Show that $\mathbb{C}P^2$ is the toric manifold whose moment polytope is the isosceles right triangle.

Solution: The moment map is

$$[z_0, z_1, z_2] \mapsto (|z_0|^2 + |z_1|^2 + |z_2|^2)^{-1}(|z_0|^2, |z_1|^2, |z_2|^2)$$

In other words, if we define

$$s_j = \frac{|z_j|^2}{\sum_{j=0}^{2} |z_j|^2},$$

the image is

$$\{(s_0, s_1, s_2)|0 \le s_j \le 1, s_0 + s_1 + s_2 = 1\}$$

This triangle is a subset of the hyperplane $\{(x, y, z)|x + y + z = 1\}$ in $\mathbb{R}^3$. We project this triangle to the (x, y) plane by the projection map $(x, y, z) \mapsto (x, y)$.

This means the moment map image is the set

$$\{(x, y)|0 \le x \le 1, 0 \le y \le 1, x + y \le 1\}$$

which is an isosceles right triangle.

Alternatively, we may define the action of $U(1) \times U(1)$ on $\mathbb{C}P^2$ by

$$(u_1, u_2) : [z_0, z_1, z_2] \mapsto [z_0, u_1 z_1, u_2 z_2].$$

Then the moment map is

$$[z_0, z_1, z_2] \mapsto (|z_0|^2 + |z_1|^2 + |z_2|^2)^{-1}(|z_1|^2, |z_2|^2)$$

The image of this moment map is

$$\{(s_1, s_2)|0 \le s_1 \le 1, 0 \le s_2 \le 1, s_1 + s_2 \le 1\}$$

with s_1, s_2 defined as above. This is the isosceles right triangle.

(3) A. Exhibit explicitly the subsets of $\mathbb{C}P^3$ for which the stabilizer under the standard action of $U(1)^3$ is

(a) isomorphic to $U(1)$

Solution: This subset is

$$\{[z_0 : z_1 : z_2 : z_3]\}$$

where one of the z_j is 0

(b) isomorphic to $U(1)^2$

Solution: This subset is

$$\{[z_0 : z_1 : z_2 : z_3]\}$$

where two of the z_j are 0.

(c) isomorphic to $U(1)^3$.

Solution: This subset is $\{[1:0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$.

B. Describe the images of these subsets under the moment map $\mu_{U(1)^3}$: show that they are

(a) facets (i.e. 2-dimensional faces)

Solution: The image is

$$\{(|z_0|^2, |z_1|^2, |z_2|^2, |z_3|^2)\}$$

(under the assumption that $\sum_i |z_i|^2 = 1$) and one of the z_j is zero). This is $\{s_0, s_1, s_2, s_3 | s_j \in [0,1], \sum_i s_i = 1\}$

The images of the facets are the subsets for which one of the s_i is 0.

(b) edges (i.e. 1-dimensional faces)

Solution: The image is as in in (a). The edges are the subsets for which two of the s_i are 0.

(c) vertices: The vertices are the subsets when three of the four s_i are 0 and the other is 1.

(4) **(Example of projection of moment polytopes)**

(a) Give an approximate sketch of the image $\mu_H(M)$ in the case when $M = \mathbb{C}P^3$, $T = U(1)^3 \subset U(1)^4$ acting in the usual way (so the moment polytope is a 3-simplex in $\mathbb{R}^4$) and $H = U(1)^2$ acting via some embedding in $U(1)^3$: choose the orthocomplement of Lie(H) in Lie(T) to be some axis $\mathbb{R}\hat{v}$ through the origin in $\mathbb{R}^3$. (The image will depend on the axis $\mathbb{R}\hat{v}$ chosen: draw the axis you have chosen.)

Solution: The image is the projection of $\mu(M) \subset \mathrm{Lie}(T)$ under the projection map from the dual of Lie(T) to the dual of Lie (H) (this is dual to the inclusion map i_H from Lie(H) to Lie(T)). To find this image, we examine the image of the set of vertices under the projection and take the convex hull.

(b) What are the *regular values* for the moment map of H?

Solution: The non-regular (in other words singular) values of this moment map are:

(i) the images under the above projection $\pi_H : \mathrm{Lie}(T) \to \mathrm{Lie}(H)$ of the images of the edges and vertices under the moment map. This is because the condition for a point f to be a regular value of a map f is that df_x is surjective for all values $x \in f^{-1}(b)$. In our case, $f = \pi \circ \mu$ where $\mu : M \to \mathfrak{t}^*$ is the moment map and $\pi : \mathfrak{t}^* \to \mathfrak{h}^*$ is the projection on $\mathfrak{h}^*$, which is orthogonal projection on the orthocomplement of $\hat{v}$. So b is a regular value provided $\pi \circ d\mu$ is surjective. This is not true when b is in an edge or is a vertex, because $d\mu$ is not surjective so likewise $\pi \circ d\mu$ is not surjective.

(ii) The orthocomplement $\hat{v}$ of $\mathfrak{h}$ lies in a face, and then all multiples of $\hat{v}$ in this face are singular values. (If b is in a face, the image of $d\mu$ is the tangent space to that face, which projects to $\mathfrak{h}$ provided that $\hat{v}$ is not in the tangent space to the face. If $\hat{v}$ is in the tangent space to the face, then $\hat{v}$ is in the kernel of the projection map so the image of the tangent space of the face under the projection map has dimension 1.)

Note that the projection of a facet may be either two-dimensional or one-dimensional, and the projection of an edge may be either one-dimensional or zero-dimensional. This depends on the choice of Lie(H) as a subset of Lie(T).

See attached figure.

$$A = (0,0,0)$$

$$B = (1,0,0)$$
$$C = (0,1,0)$$
$$D = (0,0,1)$$
$$E = (1,1,1)/3$$

In part (a), the tetrahedron which is the convex hull of A, B, C, D is projected on the plane through A, C, D. The vector $\hat{v}$ is $(1,0,0)$.

In part (b), the same tetrahedron is projected on a plane through $(0,0,0)$ which is parallel to the plane through B, C, D. The vector $\hat{v}$ is $(1,1,1)$.