MAT1312 Exercises 1

Due date: 5th week (week of Feb 6, 2012)

1. Let \( f : \mathbb{C} \to \mathbb{R} \) be the moment map for the standard action of \( U(1) \) on \( \mathbb{C} \) by rotation:

\[
f : z \mapsto -|z|^2/2
\]

Construct the Hamiltonian flow of \( f^2 \) and \( f^3 \). Show that the orbits are all periodic but of different period depending on the value of \( |z| \): find the period of the orbit as a function of the radius. (Thus the functions \( f^2 \) and \( f^3 \) are NOT moment maps for a circle action, although all orbits are periodic.)

2. (Coadjoint orbits in \( u(n) \))

Recall that any matrix in \( u(n) \) may be conjugated to a matrix of the form

\[
\text{diag}(i\lambda_1, \ldots, i\lambda_n),
\]

where the \( \lambda_j \in \mathbb{R} \) and \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \).

Let \( G = U(n) \) and let \( M \) be the orbit of \( \text{diag}(i\lambda_1, \ldots, i\lambda_n) \) in the Lie algebra \( \mathfrak{g} \) of \( U(n) \):

\[
M = \{ A \in GL(n, \mathbb{C}) : A + A^\dagger = 0, \ A \text{ has eigenvalues } i\lambda_1, \ldots, i\lambda_n \}.
\]

Let \( T \) be the maximal torus of \( U(n) \).

Show that the moment map for the action of \( T \) on \( M \) is the projection

\[
A \mapsto (A_{11}, \ldots, A_{nn})
\]

onto the diagonal elements of the matrix.

3. Let \( T \) be a torus acting on a symplectic manifold \( M \) in a Hamiltonian way, and let \( F \subset M^T \) be a component of the fixed point set. Show that \( \mu(F) \) is a point. (Hint: show that for any \( f \in F, d\mu_f = 0 \).)

4. Show that the Hamiltonian vector fields of the components of the map

\[
\mu : T^*\mathbb{R}^3 \to \mathbb{R}^3
\]

given by the cross product,

\[
\mu : (\vec{q}, \vec{p}) \mapsto \vec{q} \times \vec{p}
\]

are the vector fields \( \dot{X} \) on \( T^*\mathbb{R}^3 \) generated by the action of \( X \in \mathbb{R}^3 = \mathfrak{g} \) on \( T^*\mathbb{R}^3 \) (where \( G = SO(3) \) acts on \( \mathbb{R}^3 \) by rotations).

5. (a) Show that if a submanifold \( N \) of a symplectic manifold \( M \) is preserved by the action of an almost complex structure \( J \) on \( M \) (in other words \( N \) is an almost complex submanifold of \( M \) with respect to \( J \)) then the symplectic form restricts to a nondegenerate form on \( N \).
(b) Assume that \( f : \mathbb{C}^n \to \mathbb{C} \) is a holomorphic function. Show that if 0 is a regular value of \( f \) then \( f^{-1}(0) \) is a symplectic submanifold of \( \mathbb{C}^n \).

(c) Assume that \( f : \mathbb{C}^n \to \mathbb{C} \) is a \textit{homogeneous} polynomial function (in other words \( f(\lambda z) = \lambda^d f(z) \) \( \forall \lambda \in \mathbb{C}^* \)). Show that if 0 is a regular value of \( f \) then \( \{ [z_1 : \ldots : z_n] \in \mathbb{C}P^{n-1} : (z_1, \ldots, z_n) \in f^{-1}(0) \} \) is a symplectic submanifold of \( \mathbb{C}P^{n-1} \).

6. (Orbits of Hamiltonian group actions are isotropic) Let \( M \) be equipped with the Hamiltonian action of a compact Lie group \( G \). Show that the orbits of the action of \( G \) on \( \mu^{-1}(0) \) are isotropic with respect to the symplectic structure.

7. (Symplectic slices) Let \( Y \) be a symplectic manifold equipped with the Hamiltonian action of a torus \( T \) which is the maximal torus of a compact Lie group \( G \) with moment map \( \mu_T : Y \to t^* \).

Define
\[
M := Y \times_T G = \{ (y, g) \in Y \times G : (y, g) \simeq (ty, tg) \text{ for } t \in T \}.
\]

Define a symplectic structure on \( M \) on with respect to which the action of \( G \) is Hamiltonian. Exhibit a moment map \( \mu_G : M \to g^* \) for the action of \( G \) on \( M \). What is \( \mu_G^{-1}(t) \)?

2) \( \text{su}(2) \)

8. Show explicitly that the diagonal elements of matrices conjugate (under \( SU(2) \)) to \( \text{diag}(2\pi i, -2\pi i) \) in \( \text{su}(2) \) are of the form \( \theta \text{diag}(2\pi i, -2\pi i) \) where \( \theta \in [-1, 1] \).