MAT415: HOMEWORK SET #3

DUE: MONDAY MARCH 9, 2015

(1) Let L be a Galois extension of K, where K, L are number fields. Let G be the galois group of L/K.

  (a) Let Q be a non-zero prime ideal of $\mathcal{O}_L$. Prove that there is a unique non-zero ideal I of $\mathcal{O}_K$ such that $I\mathcal{O}_L = \prod_{g\in G} g(Q)$. Define $\mathrm{Norm}_{L/K}Q = I$.

  (b) In the above situation, find an example of K, L, Q such that $\mathrm{Norm}_{L/K}Q$ is NOT a prime ideal.

  (c) Prove that for each non-zero fractional ideal J of L, there is a unique non-zero ideal I of $\mathcal{O}_K$ such that $I\mathcal{O}_L = \prod_{g\in G} g(J)$. Define $\mathrm{Norm}_{L/K}J = I$.

  (d) Prove that $\mathrm{Norm}_{L/K}$ is a group homomorphism from I_L to I_K.

  (e) For $\alpha \in L^\times$, prove that $\mathrm{Norm}_{L/K}(\alpha\mathcal{O}_L) = (\mathrm{Norm}_{L/K}\alpha)\mathcal{O}_K$.

  (f) Prove that $\mathrm{Norm}_{L/K}(I)$ is generated as an $\mathcal{O}_K$ ideal by $\mathrm{Norm}_{L/K}\alpha$ where α varies over I.

(2) Let $K \subset L$ be number fields, and let M be a Galois extension of K containing L, such that M is also a number field.

  (a) For a non-zero fractional ideal J of L, prove that there is a unique non-zero fractional ideal I of K such that
$$I\mathcal{O}_M = \prod_{g\in \mathrm{Gal}(M/K)/\mathrm{Gal}(M/L)} g(J\mathcal{O}_M).$$
  Define $\mathrm{Norm}_{L/K}J = I$.

  (b) Prove that $\mathrm{Norm}_{L/K}J$ is independent of the choice of M.

  (c) Prove that $\mathrm{Norm}_{L/K}$ is a group homomorphism from I_L to I_K.

  (d) For $\alpha \in L^\times$, prove that $\mathrm{Norm}_{L/K}(\alpha\mathcal{O}_L) = (\mathrm{Norm}_{L/K}\alpha)\mathcal{O}_K$.

(3) This exercise is designed to show that the discriminant of a number field is always 0 or 1 modulo 4. Let K be a number field and $\alpha_1, \ldots, \alpha_n$ a $\mathbb{Z}$-basis for $\mathcal{O}_K$. Recall that the discriminant D_K of K is the square of the determinant of M, where M is the $n \times n$ matrix given by $M_{ij} = \sigma_i(\alpha_j)$, where $\sigma_1, \ldots, \sigma_n$ are the n embeddings of K into $\mathbb{C}$. For a permutation τ of $\{1, \ldots, n\}$ recall that τ is even if it is in the alternating group, and odd otherwise.

Set

$$A = \sum_{\tau \text{ even}} \sigma_i(\alpha_{\tau(i)}), B = \sum_{\tau \text{ odd}} \sigma_i(\alpha_{\tau(i)}).$$

By expanding the determinant, we see that $D_K = (A - B)^2$.

(a) Prove that A, B are both contained in a quadratic field k, and that $A + B$ is an integer.

(b) Conclude that D_K is always 0 or 1 modulo 4.

(4) Let K be a number field. This exercise will study modules over $\mathcal{O}_K$:

(a) Prove that for each positive integer n, $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is a principal ideal domain.

(b) Prove that every finitely generated torsion module M over $\mathcal{O}_K$ is a direct sum of modules of the form $\mathcal{O}_K/P^e$ where P is a non-zero prime ideal of $\mathcal{O}_K$ and e is a positive integer.

(c) Let I and J be non-zero fractional ideals of K. Prove that I and J are equivalent in the class group of K iff I and J are isomorphic as $\mathcal{O}_K$-modules.

(d) Prove that $I \otimes_{\mathcal{O}_K} J$ and IJ are naturally isomorphic as $\mathcal{O}_K$ modules. **Hint: There is a natural map $I \otimes_{\mathcal{O}_K} J \to IJ$. Prove that this map is an isomorphism of abelian groups when tensored over $\mathbb{Z}/n\mathbb{Z}$ for any positive integer n.**

(e) Prove that any non-zero fractional ideal I of $\mathcal{O}_K$ is projective as an $\mathcal{O}_K$ module. That is, I is a direct summand of a free module. Equivalently, for every surjection of $\mathcal{O}_K$ modules $\phi : A \to B$ and every map of $\mathcal{O}_K$-modules $\psi : I \to B$, there is map $\tilde{\psi} : I \to A$ such that $\phi \circ \tilde{\psi} = \psi$. **Hint: Prove there is a (necessarily split) surjection of $\mathcal{O}_K$ modules $I^{-1} \oplus I^{-1} \to \mathcal{O}_k$, and then tensor with I.**

(f) Prove that every finitely generated torsion free $\mathcal{O}_K$-module can be written as $\bigoplus_{i=1}^{r} I_i$ for non-zero ideals I_i of $\mathcal{O}_K$.

(g) Prove that if I is an ideal of $\mathcal{O}_K$, and $\mathcal{O}_K \oplus I$ is isomorphic to $\mathcal{O}_K \oplus \mathcal{O}_K$ then I is principal.

(h) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{-3})$ and $K = \mathbb{Q}(\sqrt{-6})$. Prove that $\mathcal{O}_L$ is not free as an $\mathcal{O}_K$ module.