MAT415: MIDTERM

Each question is worth 12 points, with each part being worth an equal number of points.

You may use any facts stated in class, and standard facts about Galois theory, finite fields, group theory without proof as long as you state clearly what you are using. You may use the Minkowski bound without proof:

$$\sqrt{|D_K|} \geq \frac{n^n}{n!} \cdot \left(\frac{\pi}{4}\right)^{r_2}$$

(1) Let $K = \mathbb{Q}[x]/(x^4 + x^3 + x^2 + x + 1)$, so that x is a primitive 5'th root of unity.

(a) Using Discriminants (or otherwise), prove that $\mathcal{O}_K = \mathbb{Z}[x]$.

(b) For each prime p, depending on what p is modulo 5 figure out how many prime ideals P in $\mathcal{O}_K$ satisfy $P \cap \mathbb{Z} = (p)$.

(2) (a) Determine the class group of $\mathbb{Q}(\sqrt{-22})$.

(b) Find all integer solutions to $x^3 = y^2 + 550$. **Hint: Think in terms of ideals**

(3) Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$, and set R to be the union of $\mathcal{O}_K$ over all number fields $K \subset \overline{\mathbb{Q}}$. Thus R is the set of all algebraic integers, and we showed in class that R is a ring.

(a) Does there exist a non-zero ideal I of R such that $I \cap \mathbb{Z} = \{0\}$?
(b) Prove that every non-zero prime ideal of R is maximal.
(c) Does there exists a prime ideal P of R such that R/P is a finite field?
(d) Prove that R is not noetherian, but that every finitely-generated ideal of R is principal.