Lecture 9 - Faithfully Flat Descent

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1 Descent of morphisms

In this lecture we study the concept of 'faithfully flat' descent, which is the notion that to obtain an object on a scheme X, it is enough to give an object on a faithfully flat cover Y of X, together with 'gluing' or 'descent' data. As this is best illustrated by example, we begin by studying descent of morphisms:

Consider a scheme X, and an open covering $(U_i)_{i\in I}$ of X. Now for any scheme Z, to give a morphism $\phi : X \to Z$ it is equivalent to give morphisms $\phi_i : U_i \to Z$ which agree on intersections, which is to say that $\phi_i \mid U_i \cap U_j = \phi_j \mid U_i \cap U_j$. Noting that in the category of sets, fiber product is the same as intersection, our first theorem is a generalization of this fact to faithfully flat coverings;

Theorem 1.1. *Lt $(U_i \to X)_{i\in I}$ be a covering in X_{fl}, and Z a scheme. Suppose we have morphisms $\phi_i : U_i \to Z$ such that for all $i, j \in I$ we have*

$$\phi_i \mid U_i \times_X U_j = \phi_j \mid U_i \times U_j.$$

Then there exists a unique map $\phi : X \to Z$ such that $\phi \mid U_i = \phi_i$ for all $i \in I$.

Before we prove this theorem, we shall need the following lemma, which is a key ingredient in main incarnations of faithfully flat descent.

Lemma 1.2. *Let $\phi : A \to B$ be a faithfuly flat map of rings. Then*

$$(*)\ 0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{d_0} B \otimes_A B$$

is an exact sequence of A-modules, where $d_0(b) := b \otimes 1 - 1 \otimes b$.

Proof. Since ϕ is injective, we consider A as a subset of B. We proceed in three stages:

1. Suppose that there exists a section $g : B \to A$ such that $g \mid A = Id_A$. Then we can write $B = A \oplus I$ where $I = \ker g$ is a B-module. Then

   $$B \otimes_A B = (A \otimes_A A) \oplus (A \otimes_A I) \oplus (I \otimes_A A) \oplus (I \otimes_A I)$$

   and for $i \in I$ we have $d^0(i) = i \otimes 1 - 1 \otimes i$. Now, since d_0 all of A in its kernel, it follows that $A = \ker d_0$, as desired.

   Alternative proof: Consider the map $h := g \otimes id : B \otimes_A B \to B$. Then $d_0(b) = 0$ implies that $0 = h \circ d_0(b) = b - g(b)$, which shows $b = g(b) \in A$.

2. Suppose that $A \to C$ is a faithfully flat extension. Then

   $$(B \otimes_A B) \otimes_A C \cong (B \otimes_A C) \otimes_C (B \otimes_A C).$$

   Thus, if we tensor (*) with C over A, we obtain

   $$0 \longrightarrow C \xrightarrow{\phi} B \otimes_A C \xrightarrow{d_0} (B \otimes_A C) \otimes_C (B \otimes_A C)\ .$$

   Since C is faithfully flat, it suffices to prove that this latter sequence is exact. Thus, we can replace the pair (A, B) with $(C, B \otimes_A C)$.

3. Finally, consider an arbitrary $A \to B$. Now we apply the previous reduction with $C = B$. Then we get the pair or rings

   $$B \hookrightarrow B \otimes_A B, b \to b \otimes 1,$$

   and we can construct a section by setting $g(b \otimes b') = bb'$. But this puts us in case (1), which completes the proof.

□

In fact, using the same proof method, one can prove the following fairly vast generalization of the previous lemma:

Lemma 1.3. *Let $\phi : A \to B$ be a faithfuly flat map of rings, and let M be any $A-module$. Then*

$$0 \longrightarrow M \longrightarrow M \otimes_A B \xrightarrow{d_0} M \otimes_A B^{\otimes 2} \longrightarrow \cdots \xrightarrow{d_{r-2}} M \otimes_A B^{\otimes r}$$

is an exact sequence of A-modules, where

$$B^{\otimes r} = B \otimes_A \otimes \cdots \otimes_A B,\ r \text{ times}$$
$$d_{r-1}(b) = \sum_i -1^i e_i$$
$$e_i(b_0 \otimes \cdots \otimes b_{r-1}) = b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1}$$

Proof. That the composition of any two maps is 0 is straightforward. To check exactness, we reduce as before to the case where there exists a section $g : B \to A$. Then consider the morphism $k_r : B^{\otimes r+2} \to B^{\otimes r+1}$ defined as

$$k_r(b_0 \otimes \cdots \otimes b_{r+1}) = g(b_0) \cdot b_1 \otimes b_2 \otimes \cdots \otimes b_{r+1}.$$

It is easily seen that $k_{r+1} \circ d_{r+1} + d_r \circ k_r = id$, from which the exactness follows.

□

Proof of theorem 1.1. :

First, by setting $Y = \sqcup_{i \in I} U_i$ we can reduce to the case of a single flat, surjective, locally of finite type map $\phi : Y \to X$ Suppose $h : Y \to Z$ is a morphism such that $h \circ p_1 = h \circ p_2$, where p_i is the map from $Y \times_X Y$ to Y by projecting onto the i'th co-ordinate. We wish to prove the existence and uniqueness of a morphism $g : X \to Z$ such that $g \circ \phi = h$.

1. We first prove that there is at most one such map g, so suppose g_1, g_2 are two such maps. Since ϕ is surjective as a map of topological spaces, it follows that g_1, g_2 agree as maps of topological spaces. Hence, let $x \in X, z = g_1(x) = g_2(x) \in Z$ and pick $y \in Y$ such that $\phi(y) = X$. Then we have induced maps

$$\mathcal{O}_{Z,z} \rightrightarrows \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$$

   where the first pair of maps is $g_1^\#, g_2^\#$ and the second map is $\phi^\#$ and the compositions agree and equal $h^\#$. Since Y is flat over X and local flat maps are faithfully flat by lecture 3, it follows that $\phi^\#$ is injective, and thus $g_1^\# = g_2^\#$. Thus g_1, g_2 induce the same maps of stalks at every point, and hence they are the same.

2. By the uniqueness above, it suffices to work locally on X. Thus take $x \in X$, and consider $y \in Y$ such that $\phi(y) = x$ and $h(y) = z$. Now let

Z' be an affine open neighborhood of z, let $Y' = h^{-1}(Z')$ and consider $\phi(h^{-1}(Z')) \subset X$. This is open, as ϕ is an open map, so let X' be an affine open neighborhood contained within it. Now as we can work locally on X, replace X, Y, Z by X', Y', Z'. Thus we can reduce to the case where X, Z are affine.

3. Write $Y = \cup_{i \in I} V_i$ where V_i is affine open. Since X is affine and quasi-compact, there is a finite subset $K \subset I$ such that

$$X = \cup_{k \in K} \phi(V_k).$$

Let J be any finite subset of I containing K, and write $Y_J := \sqcup_{j \in J} V_j$. This is affine, so we can write $Y_J = \text{Spec } B$. Likewise, let $X = \text{Spec } A$ and $Z = \text{Spec } C$. Now, the sequence

$$\text{Hom}(X, Z) \to \text{Hom}(Y_J, Z) \rightrightarrows Hom_(Y_J \times_X Y_J, Z)$$

can be rewritten

$$\text{hom}(C, A) \to \text{hom}(C, B) \rightrightarrows \text{hom}(C, B \otimes_A B).$$

Since the hom functor is always left-exact, that this sequence is exact follows from lemma 1.2.

Thus, there exists a unique map $g : X \to Z$ inducing $h : Y_J \to Z$. Since J was an arbitrary finite subset and g is unique, it follows that $g \circ \phi = h$, as desired.

□

Excercise 1.4. *Let X, Z be schemes. Prove that $\mathfrak{F}_Z(U) := \text{hom}(U, Z)$ is a sheaf on X_{fl}.*

2 Descent of Modules and Affine Schemes

Let $A \to B$ be a faithfully flat morphism of rings, and let M be an A-module. Then $M' := M \otimes_A B$ is a B-module. Moreover, we can define two $B \otimes_A B$ modules, given by $M' \otimes_A B$ and $B \otimes_A M'$. Note that while the underlying sets of these two modules are clearly the same, the actions of $B \otimes_A B$ are very different. Nontheless, in this case we have an isomorphism $\phi_M : B \otimes_A M' \cong M' \otimes_A B$ given by

$$\phi_M(b \otimes (m \otimes b')) = (m \otimes b) \otimes b'.$$

This induces three morphisms

$$\begin{aligned}
\phi_{M,2,3} &: B \otimes_A B \otimes_A M' \to B \otimes_A M' \otimes_A B \\
\phi_{M,1,3} &: B \otimes_A B \otimes_A M' \to M' \otimes_A B \otimes_A B \\
\phi_{M,1,2} &: B \otimes_A M' \otimes_A B \to M' \otimes_A B \otimes_A B
\end{aligned}$$

by setting $\phi_{i,j}$ to be induced by applying ϕ_M on the i, j'th co-ordinates, and one can check that $\phi_{M,1,3} = \phi_{M,1,2} \circ \phi_{M,2,3}$.

Remark: One can think of this as a cocycle condition in the following way: Consider M' as a sheaf of modules on Spec B. Recall that to give a sheaf on X from sheafs S_i on U_i where $(U_i)_{i \in I}$ is an open covering of X, one needs to give isomophisms $\phi_{i,j} : S_i \mid U_i \cap U_j \cong S_j \mid U_i \cap U_j$ together with the gluing condition $\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}$. Now thinking of Spec B as a covering of A, we need to check the same conditions. This looks a little weird as we only have one open set, but becomes non-trivial since our morphisms are now more complicated. Thus, the analogue in this setting is an isomorphism

$$\phi_{1,2} : p_1^* M \to p_2^* M$$

where p_1, p_2 are the projection maps on Spec $B \times_{\text{Spec } A}$ Spec B, together with the gluing condition $\phi_{1,3} = \phi_{2,3} \circ \phi_{1,2}$. This is precisely what our definition reflects. This discussion the following lemma more intuitive, and justifies our treating faithfully flat morphisms as coverings.

Theorem 2.1. *Every pair (M', ϕ) where $\phi : B \otimes_A M' \cong M' \otimes_A B$ satisfies $\phi_{1,3} = \phi_{1,2} \circ \phi_{2,3}$ there is an A-module M, unique up to isomorphism, with a unique isomorphism $M' \cong M \otimes_A B$ identifying ϕ with ϕ_M.*

Remark. *Note that the condition that ϕ be identified with ϕ_M is crucial for the uniqueness statement, and this will in turn be crucial for the gluing argument once we generalize this to schemes.*

Proof. Define the A-module M by setting $M = \{m \in M' \mid \phi(1 \otimes m) = m \otimes 1\}$. We claim that the natural map $M \otimes_A B \to M'$ is an isomorphism, which moreover identifies ϕ_M with ϕ. To prove this, consider the following diagram:

$$\begin{array}{ccc}
B \otimes_A M' & \overset{1\otimes\alpha}{\underset{1\otimes\beta}{\rightrightarrows}} & B \otimes_A M' \otimes_A B \\
\downarrow \phi & & \downarrow \phi_{1,2} \\
M' \otimes_A B & \overset{1\otimes\pi_2}{\underset{1\otimes\pi_3}{\rightrightarrows}} & M' \otimes_A B \otimes_A B
\end{array}$$

where $\alpha(m) = m \otimes 1$ and $\beta(m) = \phi(1 \otimes m)$. The cocycle condition implies that the diagram commutes with either the top or bottom arrows, hence the left downward map ϕ identifies their kernels. Since B is faithfully flat over A, the upper kernel is $B \otimes M$ and by lemma 1.3 the bottom kernel is M'. This proves the claim.

□

We are now ready to prove a descent theorem for schemes:

Theorem 2.2. *Let $Y \to X$ be faithfully flat. Suppose $Z' \to Y$ is an* affine *scheme, and $\phi : Y \times_X Z' \to Z' \times_X Y$ is an isomorphism of $Y \times_X Y$ schemes, such that $\phi_{1,3}$ and $\phi_{2,3} \circ \phi_{1,2}$ induce the same isomorphism*

$$Y \times_X Y \times_X Z' \to Z' \times_X Y \times_X Y.$$

Then up to isomorphism, there exists a unique pair (Z, ψ) consisting of an affine X-scheme $Z \to X$ and an isomorphism

$$\psi : Z \times_X Y \to Y$$

of Y-schemes, such that under the identification ψ, the map ϕ on $Y \times_X Z \times_X Y$ becomes the natural map $(y_1, z, y_2) \to (z, y_1, y_2)$.

Proof. By the uniqueness claim, we may work locally on X and Y, wlog $X = \text{Spec } A, Y = \text{Spec } B, Z' = \text{Spec } C$. But then the theorem follows immediately from lemma 2.1 if we let M' be the B-module C.

□

Excercise 2.3. *Let $\phi : Z \to X$ be a morphism, and $\psi : Y \to X$ be a faithfully flat morphism, and $\phi_Y : Z \times_X Y \to X$ the base change of ϕ along ψ. For each of the following properties P, prove that ϕ is a morphism of type P iff ϕ_Y is:*

- *Open Immersion*
- *Unramified*
- *Finite Type*
- *Finite*
- *Flat*
- *Etale*
- *Faithfully Flat*
- *Closed Immersion*

3 Locally constant sheaves

Let X be a connected scheme.

Definition. *A sheaf $\mathfrak{F}$ of sets(resp. abelian groups) on X_E is* constant *if there exits a set(resp. abelian group) S such that $\mathfrak{F}(U) \cong S^{\pi_0(U)}$, where $\pi_0(U)$ denotes the set of connected components of U. We define the* rank *if $\mathfrak{F}$ to be the cardinality of S. A sheaf $\mathfrak{F}$ is* locally constant *if there exists a covering $(U_i \to X)_{i \in I}$ such that the restriction of $\mathfrak{F}$ to $(U_i)_E$ is constant.*

It is not difficult to see that for any two U_i with non-empty fiber product the rank of $\mathfrak{F}$ on $(U_i)_E$ is the same. It thus follows since X is connected that we have a well-defined notion of rank for locally constant sheaves.

Lemma 3.1. *Let E be et or fl. If $Z \to X$ is a finite etale cover, then $\mathfrak{F}_Z(U) := Hom_X(U, Z)$ defines a locally constant sheaf. Moreover, if Z' is another finite etale cover of X, then* $\hom(\mathfrak{F}_Z, \mathfrak{F}_{Z'}) \cong \hom_X(Z, Z')$.

Proof. That $\mathfrak{F}_Z$ is a sheaf does not use that Z is finite etale, and follows immediately from theorem 1.1 (see Ex. 1.4). Now, it is easy to see that for any E-morphism $Y \to X$ the restriction of $\mathfrak{F}_Z$ to Y_E is $\mathfrak{F}_{Z \times_X Y}$. Consider a galois cover $Y \to X$ such that Y surjects onto Z. By the equivalence of categories between finite etale covers and $\pi_1(X)$ sets proven in lecture 6, it follows that $Y \times_X Z$ as a scheme over Y is isomorphic to $Y^{\hom_X(Y,Z)}$. Thus, $\mathfrak{F}_Z$ restricted to Y_E is the locally constant sheaf corresponding to the set $S = \hom_X(Y, Z)$.

For the second part of the lemma, consider $\phi \in \hom(\mathfrak{F}_Z, \mathfrak{F}_{Z'})$. Then $\phi_Z : \mathfrak{F}_Z(Z) \to \mathfrak{F}_{Z'}(Z)$, so $\phi_Z(1_Z) \in \hom(Z, Z')$.

Conversely, for $\psi \in \hom_X(Z, Z')$ we get a map $\mathfrak{F}_Z \to \mathfrak{F}_{Z'}$ via composition with ψ. It is straightforward that these two maps are inverses to each other, which proves the claim.

□

Theorem 3.2. *The functor $Z \to \mathfrak{F}_Z$ defines an equivalence of categories between finite etale covers of X and locally constant sheaves of finite rank.*

Proof. In view of lemma 3.1 it suffices to prove that an arbitrary locally free sheaf $\mathfrak{F}$ of finite rank is of the form $\mathfrak{F}_Z$ for some Z. Since $\mathfrak{F}$ is locally free, it follows that there exists a covering $(U_i \to X)_{i \in I}$ such that the restriction of $\mathfrak{F}$ to $(U_i)_E$ is constant. Let $Y = \bigcup_{i \in I} U_i$. Then $\mathfrak{F}$ is constant on Y_E - associated to some finite set S - so there exists a scheme $Z' = Y^S$ which is finite etale over Y and an isomorphism $\phi : \mathfrak{F}_{Z'} \cong \mathfrak{F} \mid Y_E$. Now, the restriction of $\mathfrak{F}$ to $(Y \times_X Y)_E$ can be thus identified with either $\mathfrak{F}_{Z' \times_X Y}$ or $\mathfrak{F}_{Y \times_X Z'}$, so we get an isomorphism of $Y \times_X Y$-scheme $\phi : Y \times_X Z' \to Z' \times_X Y$. Moreover, restricting to $Y \times_X Y \times_X Y$ shows that $\phi_{1,3} = \phi_{2,3} \circ \phi_{1,2}$. Thus by theorem 2.2 we get a scheme $Z \to X$ and an identification $\psi : Z \times_X Y \to Z'$ which makes ϕ into the natural 'flip the co-ordinates' map. Now theorem 1.1 together with the sheaf condition for $\mathfrak{F}$ gives us an identification $\mathfrak{F} \cong \mathfrak{F}_Z$. Finally, that Z is itself etale follows from excercise 2.3.

□

As an immediate corollary to theorem 3.2 together with the equivalence of categories between finite etale covers and $\pi_1(X)$ sets proven in lecture 6, we have the following

Corollary 3.3. *Let $\overline{x}$ be a geometric point of X. The category of locally constant sheaves of sets(resp. abelian groups) is naturally isomorphic to the category of $\pi_1(X, \bar{x})$-sets(resp. modules).*