Lecture 8 - Sites, Presheaves and Sheaves

October 15, 2014

1 Sites

As we described in the first lecture, in order to obtain a topology on schemes to facilitate a Weil Cohomology theory, we need to look beyond the Zariski topology. In fact, what Grothendieck realized is that to define an interesting theory of sheaves and their cohomology, we needn't bother with ordinary topology at all; one can define what's called a 'site' or a 'Grothendieck Topology', which provides a much more general framework as follows:

Definition. *As site S consists of the following data:*

1. *A category C admitting fiber products*
2. *A class[1] E of families of morphisms in C, $(U_i \to U)_{i \in I}$ which we shall term 'coverings', such that*
   - *Coverings are closed under base change. That is, given a covering $(U_i \to U)_{i \in I}$ and a morphism $V \to U$, the family $(U_i \times_U V \to V)$ is also a covering*
   - *Coverings are closed under composition. That is given a covering $(U_i \to U)_{i \in I}$ and for each $i \in I$, a covering $(U_{i,j} \to U_i)_{j \in J_i}$ the family $(U_{i,j} \to U)_{i \in I, j \in J_i}$ is also a covering.*
   - *The family consisting of the single map $(U \to V)$ is a covering whenever $U \to V$ is an isomorphism.*

Examples:

1. Let X be a topological space. Define X_{top} to be the following site : The underlying category has as its objects the open subsets of X, and

[1] In these notes, we shall completely ignore all set-theoretic issues.

the morphisms are given by inclusion. I.e. if U, V are two open sets, then

$$\mathrm{Mor}(U,V) = \begin{cases} \emptyset & U \not\subset V \\ \{U \hookrightarrow V\}^2 & U \subset V \end{cases}$$

We say that a family $(U_i \to U)_{i \in I}$ of inclusions is a covering if $\bigcup_{i \in I} U_i = U$. This examples demonstrates that sites are a strict generalization of topological spaces. We call this the small topological site over X.

2. Let C be the site whose underlying category is the category of sets, and such that $(\phi_i : U_i \to U)_{i \in I}$ is a covering if $\bigcup_{i \in I} \phi_i(U_i) = U$.

3. For any category C, we have the site C_{triv} with C as the underlying category, and such that ALL families of maps are coverings.

4. Let X be a topological space. We define Top/X to be the category of topological spaces together with a continuous 'structure morphism' to X, so that our objects are pairs $(Y, \phi : Y \to X)$, and morphisms are continuous maps which commute with the structure morphism. Then we define $(Top/X)_{top}$ as the site whose underlying category is Top/X and such that $(\phi : U_i \to U)_{i \in I}$ is a covering if $\bigcup_{i \in I} \phi_i(U_i) = U$. We call this the big topological site over X.

Exercise: For each of the above examples, prove that we really have defined a site. That is, that coverings are closed under base change and composition.

We now define a family of sites which pertain to schemes. Let E be a class of morphisms satisfying the following conditions:

- E contains all isomorphisms
- The composition of two morphisms in E is in E.
- any base change of a morphism in E is in E.

There are three examples of such classes with which we shall be chiefly concerned

- The class $E = Zar$ of open immersions.
- The class $E = et$ of etale morphisms.

- The class $E = fl$ of flat morphisms which are locally of finite type.

A morphism which is in E shall be referred to as a *E-morphism.* Recall that SCH/X denotes the category of X-schemes, and let E/X denote the full subcategory of X-schemes whose structure morphism is in E. That is, maps $\phi : Y \to X$ such that $\phi \in E$. Note that we do not require morphisms in E/X to be in E. We say that a family $(\phi_i : U_i \to Y)_{i \in I}$ is an E-covering if all the maps are E-morphisms are $\bigcup_{i \in I} \phi_i(U_i) = U$.

Definition. *The big E-site $(SCH/X)_E$ has SCH/X as its underlying category, and the coverings are E-coverings. The small E-site $(E/X)_E$ has E/X as its underlying category, and the coverings are E-coverings.*

Note that the big E-site contains many more objects than that small E-site. For instance, the small etale site over a field has only finite unions of field spectra as its objects, whereas the big etale site contains ALL schemes over that field. A useful analogy to keep in mind is like a topological space whereas a big site is like the category of all topological spaces.

As a matter of notation, we shall write X_{zar} and X_{et} for the small etale and Zariski sites respectively, whereas we shall write X_{fl} for the big flat site[3]

2 Presheaves and Sheaves

The definition of a site allows us to carry over most of the usual structure of sheaves and presheaves with very little change

Definition. *Let S be a site with underlying category C. A presheaf of (sets,abelian groups,rings) P on S is a contravariant functor from C to the category of (sets,resp. abelian groups, resp. rings). That is, to each object U of C we assign a (set,resp. abelian group, resp. ring) $P(U)$ and for each morphism $V \to U$ a homomorphism $Res_{U,V} : P(V) \to P(U)$ such that if $W \to V \to U$ then $Res_{U,W} = Res_{V,W} \circ Res_{U,V}$. We call the maps $Res_{U,V}$* restriction *maps.*

A morphism between two presheaves $\phi : P \to P'$ consists of maps $\phi_U : P(U) \to P'(U)$ for each object U, which are compatible with the restriction maps in the sense that $\phi_V \circ Res_{U,V} = Res_{U,V} \circ \phi_U$.

[3] This notation comes about as these are the sites most often used in practise. The fact that it is logically absurd is no deterrent!

As a matter of notations, if $V \to U$ and $s \in P(U)$, we frequently write s_V for $Res_{U,V}(s)$. Note that this notation suppresses which morphism $V \to U$ we are restricting with respect to, but this should always be clear from context.

Examples:

- Given any set S, we can define a presheaf P_S on X_{fl} by $P_S(U) := S^{\pi_0(U)}$, where $\pi_0(U)$ is the set of connected components of U, with the transition maps arising from the maps on π_0.
- Given any set S, we can define a presheaf P on C by $P(U) := S$ for all objects U, with al the transition maps being the identity morphisms.
- Recall that for any topological space X with a sheaf $\mathfrak{F}$ on it, $\Gamma(X, \mathfrak{F})$ denotes the global sections of $\mathfrak{F}$ on X. Then the presheaf $P(U) := \Gamma(U, \mathcal{O}_U)$ defines a presheaf on X_{fl}, with the restriction morphisms arising from the pullback maps.

Note that the definition of a presheaf has nothing at all to do with the covering morphisms; it depends purely on the underlying category of the site. We now use the covering maps to formulate 'gluing' conditions in order to define a sheaf:

Definition. *A pre-sheaf $\mathfrak{F}$ on a site S is a sheaf if for all coverings $(U_i \to U)_{i\in I}$ the sequence*

$$(*)\mathfrak{F}(U) \to \prod_{i\in I} \mathfrak{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathfrak{F}(U_i \times_U U_j)$$

is exact. In other words, the following two conditions are satisfied:

- *If two sections $s, t \in \mathfrak{F}(U)$ are such that $\forall i \in I, s_{U_i} = t_{U_i}$ then $s = t$.*
- *If $s_i \in \mathfrak{F}(U_i)$ such that $\forall i, j \in I, (s_i)_{U_i \times_U U_j} = (s_j)_{U_i \times_U U_j}$ then there exists $s \in \mathfrak{F}(U)$ such that $\forall i \in I, s_{U_i} = s_i$.*

A morphism between sheaves is simply a morphism of the underlying presheaves.

Note that for a topological space X, a sheaf $\mathfrak{F}$ on the site X_{top} corresponds precisely to a sheaf on X in the usual topological sense.

Example 2.1. *As an example, consider the site X_{et}, and let $Y \to X$ be a Galois covering with automorphism group $G = \{\sigma_1, \dots, \sigma_n\}$. Now, from the equivalence of categories between finite etale covers and $\pi_1(X)$-sets proven in lecture 6, it follows that $Y \times_X Y = \cup_{i=1}^n Y_i$, where Y_i is simply the scheme Y, and its two maps to Y via π_1, π_2 are 1 and σ_i. Thus, (*) in this case becomes*

$$\mathfrak{F}(X) \longrightarrow \mathfrak{F}(Y) \underset{(\sigma_1,\dots,\sigma_n)}{\overset{(1,\dots,1)}{\rightrightarrows}} \mathfrak{F}(Y)^n \ ,$$

and thus the condition becomes that $\mathfrak{F}(X) \cong \mathfrak{F}(Y)^G$.

Now, it is usually impractical to check (*) for all coverings simply because there are so may. The following lemma allows us to check more easily whether a presheaf is a sheaf.

Lemma 2.2. *Let P be a presheaf on X_E, where E is one of fl or et. The P is sheaf iff the following two conditions hold:*

1. *For all $U \to X$ in X_E, the restriction of P to U_{zar} is a sheaf. That is, (*) holds for all coverings of U by open subsets.*
2. *(*) holds for coverings $(U' \to U)$ consisting of a single map, in which U, U' are both affine.*

Proof. The necessity is obvious, so we focus on the sufficiency. Consider a covering $(U_i \to U)_{i \in U}$ in X_E. Letting U' be the disjoint union of U_i, it follows from the conditio (1) that $P(U') = \prod_{i \in I} P(U_i)$, so that it suffices to prove $(*)$ for the covering $U' \to U$. We thus restrict to coverings consisting of a single map.

Now, consider $f : U' \to U$ and take a covering $U = \cup_{i \in I} U_i$ by open affine's, and let $f^{-1}(U_i) = \cup_{j \in J_i} U'_{i,j}$. Now, since f is flat and locally of finite type it follows from lecture 2 that f is open, thus there's a finite subset K of J_i such that $U_i = \cup_{j \in K} f(U'_{i,j})$. So by possibly introducing infinite redundancy we can write $U = \cup_{i \in I} U_i$ and $U' = \cup_{i \in I, k \in K_i} U'_{i,k}$ such that $f(U'_{i,k}) \subset U_i$ and K_i is finite for each $i \in I$. Now consider the following commutative diagram:

$$\begin{tikzcd}
P(U) \arrow[r, "\phi"] \arrow[d] & P(U') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P(U' \times_U U') \arrow[d] \\
\prod_{i\in I} P(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_{i,k} P(U'_{i,k}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_{i,k,k'} P(U'_{i,k} \times_U U'_{i,k'}) \\
\prod_{i,j} P(U_i \cap U_j) \arrow[r, "\psi"] & \prod_{i,j,k_i,k_j} P(U'_{i,k_i} \cap U'_{j,k_j}) &
\end{tikzcd}.$$

Since finite unions of affine schemes are affine, condition 2 says that Row 2 is exact. Moreover, condition 1 says that Columns 1 and 2 are exact. Now, a diagram chase says that ϕ is exact. But as ϕ is an arbitrary map of affine schemes, it follows that ψ is also injective. A diagram chase now says shows that Row 1 is exact, as desired. We leave the diagram chases as (useful!) exercises to the reader.

$\square$

Exercise: Let C be the site whose underlying category is the category of sets, and such that coverings are families $(\phi_i : U_i \to U)_{i\in I}$ such that $\bigcup_{i\in I} \phi_i(U_i) = U$. Prove that there is an equivalence of categories between the category of sheaves of sets(resp. abelian groups) on this site and the category of sets(resp. abelian groups).

3 Sheaves on the site (Spec $k)_{et}$

Let k be a field. The aim of this section is to describe sheaves on the site (Spec $k)_{et}$. We give a slightly different presentation than [Milne, EC, pp.52-52], so the reader may want to compare.

Our starting point is that by Lecture 6, there is an equivalence of categories between finite etale covers of Spec k and finite, discrete G-sets, where G denotes the absolute Galois group of k. Now, any etale cover of Spec k is a disjoint union of spectra of separable field-extensions of k, and in particular are locally finite covers. Thus, it follows that the underlying category (Spec $k)_{et}$ is equivalent to the category G_k-set of discrete G_k-sets. Moreover, under this equivalence a family $(\phi_i : M_i \to M)_{i\in I}$ is a covering iff $\cup_i \phi_i(M_i) = M$. We denote this equivalent category by $(G - sets)$.

Now suppose $\mathfrak{F}$ is a sheaf on $(G - sets)$. Write $G = \varprojlim G_i$ as a limit of its finite quotients. Now, G acts on G_i for each i via right multiplication, and sice $\mathfrak{F}$ is a contravariant functor we get an induced left action of G on $\mathfrak{F}$ via $g \to Res_g$. Since these actions are compatible, we can an induced action of G on $S_{\mathfrak{F}} := \varinjlim \mathfrak{F}(G_i)$, making $S_{\mathfrak{F}}$ a discrete G-module.

Conversely, suppose that S is any discrete G-module. We can define a sheaf $\mathfrak{F}_S$, by setting $\mathfrak{F}_S(M) := \mathrm{Hom}_G(M, S)$ for any discrete G-module M. Clearly, this defines a pre-sheaf. To check the sheaf condition (*) in this case is also easy, and we leave it as an exercise to the reader.

Theorem 3.1. *$S \to \mathfrak{F}_S$ and $\mathfrak{F} \to S_{\mathfrak{F}}$ are inverse functors, and define equivalences between the category of sheaves on $(G - sets)$ and the category of discrete G-sets.*

Proof. By definition, $S_{\mathfrak{F}_S} := \lim \mathfrak{F}_S(G_i) = \mathrm{Hom}_G(G_i, S)$. Hence, since S is discrete the natural map $S \to S_{\mathfrak{F}_S}$ given by $s \to \phi_s$ where $\phi_s(1) = s$ is an isomorphism.

Moreover, by example 2.1 we have that $\mathfrak{F}(M) \cong \mathrm{Hom}_G(M, S_{\mathfrak{F}})$, which shows that the natural map $\mathfrak{F} \to \mathfrak{F}_{S_{\mathfrak{F}}}$ is an isomorphism. This completes the proof.

□