1 Review of the topological fundamental group and covering spaces

1.1 Topological fundamental group

Suppose $X$ is a path-connected topological space, and $x \in X$. Then we can define a group $\pi_1(X,x)$ as follows: An element consists of a continuous map $\gamma : \mathbb{R}/\mathbb{Z} \to X$ such that $\gamma(0) = x$, which one should think of as a ‘loop based at $x$’, up to homotopy, so that two paths $\gamma_1, \gamma_2$ are considered identical if there is a continuous map $F : (\mathbb{R}/\mathbb{Z}) \times [0,1] \times X$ sending all of $0 \times [0,1]$ to $x$ such that $F_0 = \gamma_1, F_1 = \gamma_2$. The multiplication is given by concatenation, so that

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} 
\gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\
\gamma_2(2t) & \frac{1}{2} < t < 1.
\end{cases}$$

In other words, we first follow $\gamma_2$, and then $\gamma_1$.

This is referred to as the fundamental group of $X$, though of course we made a choice of a point $x \in X$. However, if $y \in X$ is any other point, and $p : [0,1] \to X$ is such that $p(0) = x, p(1) = y$, then there is an isomorphism $p : \pi_1(X,x) \to \pi_1(X,y)$ given by — using informal notation — $\gamma \mapsto p \cdot \gamma \cdot p^{-1}$.

In other words, we follow $p$ backwards to $x$, then follow $\gamma$, then follow $p$ back to $y$. Thus, the group $\pi_1(X,x)$ is - up to isomorphism - independent of $x$.

If $\phi : Y \to X$ is a continuous map between path-connected spaces, and $\phi(y) = x$ then we get an induced homomorphism $\pi_1(Y,y) \to \pi_1(X,x)$.

1.2 Covering spaces and main theorem

We say that $\phi : Y \to X$ is a covering space, if each point $x \in X$ has an open neighborhood $U$ such that $\phi^{-1}(U)$ is homeomorphism to $U \times I$ for a discrete set $I$. It is easy to see that if $X$ is connected, then $|\phi^{-1}(x)|$ is
independent of \( x \), and we call it the degree of \( \phi \), written \( \deg \phi \). Given such a space, we get an action of \( \pi_1(X, x) \) on by defining \( \gamma \) as follows: Consider \( \gamma \) as a function \( \gamma : [0, 1] \to X \), and define \( \gamma_y : [0, 1] \to Y \) to be the lift of \( \gamma \) such that \( \gamma(0) = y \). Then we define \( \gamma \cdot y \) to be \( \gamma_y(1) \). This gives us a functor \( F \) from covering spaces over \( X \) to \( \pi_1(X, x) \)-sets.

Then we have the following theorem:

**Theorem 1.1.** Suppose \( X \) is a path-connected, locally simply connected space. Then

\[
F : \{ \text{Covering Spaces over } X \} \to \{ \pi_1(X, x) - \text{sets} \}
\]

is an equivalence of categories.

Note that it follows that there exists a ‘universal covering space’ \( \tilde{X} \) corresponding to the \( \pi_1(X, x) \) viewed as a \( \pi_1(X, x) \)-set via the natural left action, whose automorphism group is naturally isomorphic to \( \pi_1(X, x) \).

### 1.3 Recovering \( G \) from the category of \( G \)-sets

Finally, we describe how - for path connected, locally simply connected \( X \)-to recover \( \pi_1(X, x) \) purely in terms of covering spaces, without mentioning loops at all. In fact, this amounts to a purely algebraic fact about groups:

Let \( G \) be a group, and consider the forgetful functor

\[
F : \{ G - \text{sets} \} \to \text{sets}
\]

which takes a \( G \)-set \( M \), and forgets the \( G \)-action. Define the Automorphism group of \( F \), written \( \text{Aut}(F) \) to be the group of sets of elements \( (\phi_M \in \text{Aut}(F(M)))_M \) — that is, for each \( G \)-set \( M \) we assign an automorphism \( \phi_M \) of the underlying set \( F(M) \) — in a way which is compatible with morphisms. That is, if \( t : M \to N \) is a morphism of \( G \)-sets, then \( \phi_N \circ F(t) = F(t) \circ \phi_M \).

**Theorem 1.2.** For any group \( G \), \( G \cong \text{Aut}(F) \).

**Proof.** First, there is a natural homomorphism \( \psi : G \to \text{Aut}(F) \) given by \( \psi(g)_M(m) = g \cdot m \). Likewise, if we consider \( G \) as a \( G \)-set in the natural way, then for any \( g \in G \) there is a morphism of \( G \)-sets given by \( r_g(g') = g'g \). Thus, it follows that if \( \phi \in \text{Aut}(F) \), then \( \phi_g = r_g \) for a \( g = \phi_G(1_G) \). It is easy to check that \( \xi(\phi) = \phi_G(1_G) \) gives a homorphism \( \xi : \text{Aut}(F) \to G \) and that \( \xi \) is a left-inverse to \( \psi \).

It remains to prove the \( \xi \) has no kernel. So Suppose \( \phi_G(1_G) = 1_G \). Then for any \( M \), there is a natural morphism of \( G \)-sets \( t_m : G \to M \) defined by
Thus, it follows that $\phi_M \circ F(t_m) = F(t_m)$. Thus, $\phi_M(m) = m$, and so $\phi_M$ is the identity for all $M$, as desired.

\[ t_m(g) = g \cdot m. \]

2 Finite Etale Morphisms

In the algebraic setting, there is no apparent analogues of loops that one can use, but we do have a reasonably good analogue of covering spaces in finite etale morphisms!

2.1 Geometric points

Let $X$ be a connected scheme, and $\pi : \text{Spec } \mathbb{K} \to X$ be a geometric point of $X$. The following lemma justifies why geometric points in algebraic geometry are a good analogue of points in topological spaces.

Definition. Let $\phi : Y \to X$ be any morphism of schemes. Given a geometric point $\overline{y} : \text{Spec } \mathbb{K} \to Y$, we define $\phi(\overline{y})$ to be the geometric point $\phi \circ \overline{y}$. We say that $\overline{y}$ is a geometric point above $\pi$, and write $\phi^{-1}(\pi)$ for the set of all such geometric points.

Lemma 2.1. Let $\phi : Y \to X$ be finite etale. Then there is a natural number $n$ such that for all geometric points $x$ in $X$, $|\phi^{-1}(x)| = n$.

Proof. Since $\phi$ is finite flat, we know from last lecture that locally we can pick neighborhoods $\text{Spec } A$ in $X$, containing the image of $\overline{x}$, such that $\phi^{-1}(\text{Spec } A) \cong \text{Spec } B$ where $B$ is free of rank $n$ as an $A$-module. $\overline{x}$ gives a morphism $A \to \mathbb{k}$, thus $B \otimes_A \mathbb{k}$ is free of rank $n$ over $\mathbb{k}$. Moreover, since separability is preserved under base change, $B \otimes_A \mathbb{k}$ is a direct sum of fields separable over $\mathbb{k}$. It follows that $B \otimes_A \mathbb{k} \cong \mathbb{k}^n$, from which the lemma easily follows.

We denote the number $n$ in the above lemma by $[Y : X]$ and refer to it as the degree of $Y$ over $X$.

2.2 Galois Covers

Let $FET/X$ denote the category of finite etale $X$-schemes. Let $Y, Z \in FET/X$, with $\pi_Y, \pi_Z$ the morphisms to $X$, such that $Y$ is connected. Any $X$-morphism $\phi : Y \to Z$ must map $\phi_{\overline{y}} : \pi_Y^{-1}(\overline{y})$ to $\pi_Z^{-1}(\overline{y})$. Moreover, since $Z \to X$ is separated and unramified and $Y$ is connected, we know from lecture 4, cor. 2.4 that $\phi$ is determined by where $\phi_{\overline{y}}$ takes a single geometric
point. As a special case, it follows that for a connected $Y \in \text{FET}/X$, we have that $|\text{Aut}_X Y| \leq [Y : X]$.

**Definition.** We say that $Y \in \text{FET}/X$ is Galois over $X$ if $Y$ is connected and $|\text{Aut}_X Y| = [Y : X]$. Equivalently, if $Y$ is connected and the action of $\text{Aut}_X Y$ on $\pi_Y^{-1}(\overline{x})$ is transitive.

**Lemma 2.2.** For any connected $Y \in \text{FET}/X$ there exists a Galois $Z$ over $X$ which surjects onto $Y$.

**Proof.** Let $n = [Y : X]$, and consider

$$Y^{(n)} := \underbrace{Y \times_X Y \times_X \cdots \times_X Y}_n,$$

where there are $n$ terms in the fiber product. Let $\overline{y_1}, \ldots, \overline{y_n}$ be the geometric points in $\pi_Y^{-1}(\overline{x})$. Now consider the geometric point $\overline{z} := (\overline{y_1}, \ldots, \overline{y_n})$ of $Y^{(n)}$, and let $Z$ be the connected component of $Y^{(n)}$ containing $\overline{z}$. We claim that $Z$ is galois over $X$.

First, consider the morphism $\pi_{1,2} : Z \to Y \times_X Y$ given by projecting onto the first two co-ordinates. Since all our schemes are Noetherian, and $\pi_{1,2}$ is finite flat, the map $\pi_{1,2}$ is surjective onto a connected component. Since the diagonal $\Delta : Y \to Y \times_X Y$ is an open immersion (as $Y$ is etale) and its also finite (being a quotient) it follows that the diagonal is a connected component of $Y \times_X Y$. Thus, since $\pi_{1,2}(\overline{z})$ has image not contained in the diagonal, it follows that $\pi_{1,2}$ has image disjoint from the diagonal, and therefore $Z$ has no geometric points above the diagonal.

Reasoning similarly for all $\pi_{i,j}$, it follows that all geometric point in $Z$ are of the form

$$\overline{z}_\sigma = (\overline{y_{\sigma(1)}}, \ldots, \overline{y_{\sigma(n)}})$$

for some automorphism $\sigma$ of $\{1, 2, \ldots, n\}$. But there is an automorphism $\sigma$ of $Y^{(n)}$ permuting the coefficients of $Y$, which send $\overline{z}$ to $\overline{z}_\sigma$. It must also send $Z$ to $Z$, since $Z$ is the connected component containing those two points, and thereby the automorphism group of $Z$ act transitively on the geometric points of $\pi_Z^{-1}(X)$. Hence $Z$ is Galois.

Finally, consider the projection $\pi_1 : Z \to Y$. $\pi_1$ is finite, and also etale by Lecture 5, lemma 1.1. Thus $\pi_1$ is surjective.

**Lemma 2.3.** If $Y$ and $Z$ are Galois over $X$, then there exist a Galois $W$ over $X$ which surjects onto $Y$ and $Z$. 

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Proof. Take $W$ to be any connected component of $Y \times_X Z$. It is easy to see that the automorphisms of $Y \times_X Z$ act transitively on the geometric points above $\mathfrak{p}$, and thus it follows as in the proof above that $W$ is Galois. Likewise, one can show the projections from $W$ to $Y$ and $Z$ are finite and flat by Lecture 5, lemma 1.1, and thus surjective. 

2.3 Universal cover of $X$

Since we are only working with finite maps, we cannot expect an object in $FET/X$ to play the role of the universal covering space in topology. However, we can approximate it by finite objects as follows:

Definition. A universal cover $\tilde{X}$ over $X$ consists of the following data\(^1\):

- A partially ordered set $I$, which is filtered in the sense that for any two objects, there is some object less than or equal to both of them.
- For each $i \in I$ a Galois cover $X_i$ of $X$.
- For any two objects $i, j \in I$ with $i < j$ a transition morphism $\phi_{ij} : X_i \to X_j$ such that $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$.

Such that any connected $Y \in FET/X$ is covered by $X_i$ for some $i \in I$. Moreover, we say that $\tilde{X}$ is based, and we write $\tilde{\mathfrak{p}}$ for its base point if each $X_i$ is assigned a geometric point $\mathfrak{p}_i$ in a compatible way with the transition maps.

Lemma 2.4. There exists a based universal cover.

Proof. Let $(X_i, \mathfrak{p}_i)_{i \in I}$ be a set of Galois schemes over $X$ together with a geometric point over $\mathfrak{p}$, indexed by some set $I$. Now we define an order in $I$ by declaring that $i \leq j$ iff there is a morphism from $X_i$ to $X_j$, in which case we set $\phi_{ij}$ to be the morphism carrying $\mathfrak{p}_i$ to $\mathfrak{p}_j$ — note we can always ensure this as $X_j$ is Galois by composing with an automorphism of $X_j$ over $X$. It is clear that this defines a partially ordered set, and its filtered by lemma 2.3. Moreover, by lemma 2.2 every finite etale connected $Y$ is covered by some $X_i$. Thus, we have constructed a based universal cover $(\tilde{X}, \tilde{\mathfrak{p}})$. 

\(^1\)In the language of category theory, we can consider $\tilde{X}$ a pro-object in the category $FET/X$, though we will not expand on this
For any scheme $Z$ over $X$, we define $\text{hom}_X(\tilde{X}, Z)$ to be the direct limit of $\underset{\leftarrow}{\text{lim}} \text{hom}_X(X_i, Z)$.

**Lemma 2.5.** For any $Y \in \text{FET}/X$, there is a natural isomorphism $F_Y : \text{hom}_X(\tilde{X}, Y) \to \pi_Y^{-1}(\overline{\pi})$ given by $\phi \mapsto \phi(\overline{\pi})$.

**Proof.** WLOG we take $Y$ to be connected. We first prove that $F_Y$ is surjective. Let $\overline{y} \in \pi_Y^{-1}(\overline{\pi})$. Now pick $i \in I$ such that there is a map $\phi : X_i \to Y$. Now $\phi$ is etale, hence surjective, hence surjective on geometric points. Thus there exist a geometric point in $X_i$, which we may write as $g \cdot \overline{x}_i$ for $g \in \text{Aut}_X X_i$ such that $\phi(g \cdot \overline{x}_i) = \overline{y}$. Then the map $\phi \circ g$ gives an element of $\text{hom}_X(\tilde{X}, Y)$, and $F_Y(\phi \circ g) = \overline{y}$.

Now to show that $F_Y$ is injective. So let $i, j \in I$ and consider two maps $\psi_i : X_i \to Y$ and $\psi_j : X_j \to Y$ such that $\psi_i(\overline{x}_i) = \psi_j(\overline{x}_j)$. We must show that the images of $\psi_i$ and $\psi_j$ are the same in $\text{hom}_X(\tilde{X}, Y)$. Let $k \in I$ be such that $X_k$ has surjective maps $\phi_{ik}$ and $\phi_{jk}$ to $X_i$ and $X_j$, mapping $\overline{x}_k$ to $\overline{x}_i$ and $\overline{x}_j$ respectively. Then the images of $\psi_i$ and $\psi_j$ in $\text{hom}_X(X_k, Y)$ both map $\overline{x}_k$ to $\overline{y}$. Thus, the two maps must be the same, and the claim follows.

Thus we see that $\tilde{X}$ plays the analogous role in $\text{FET}/X$ to a universal cover.

### 2.4 Quotients of finite etale morphisms

To understand all of $\text{FET}/X$ from just the Galois objects, we shall need to take quotients of schemes. We begin by looking at a more general context:

**Definition.** Suppose $X$ is a scheme and $G$ is a group acting on $X$. We say that a morphism $\phi : X \to Y$ is the categorical quotient of $X$ by $G$ if for any morphism $\psi : X \to Z$ for which $\psi = \psi \circ g$ for all $g \in G$, $\psi$ factors uniquely through $\phi$. In this case, we write $Y = G \backslash X$.

Note that if group schemes exist, they are unique up to isomorphism by the universal property. It turns out that such quotients do not always exist, so one must be a little careful. However, we have the following fairly general lemma:

**Lemma 2.6.** Suppose $f : Y \to X$ is an affine morphism of schemes, and $G$ is a finite group acting on $Y$ which preserves $f$. Then $G \backslash Y$ exists.
Proof. First, note that if \((U_\alpha)\) is an open covering of \(X\) such that the quotient of \(f^{-1}(U_\alpha)\) by \(G\) exists for each \(\alpha\), then we can glue all these quotients by the uniqueness given in the universal property to create a quotient of \(Y\) by \(G\). Thus, the statement is local on \(X\), and so we may suppose \(X = \text{Spec} A\) is affine. Since \(f\) is an affine morphism, it follows by definition that \(Y = \text{Spec} B\) is affine.

We claim that \(\text{Spec } B^{G}\) is the quotient of \(\text{Spec } B\) by \(G\). To prove this, suppose \(\phi : \text{Spec } B \to Z\) is a \(G\)-invariant morphism. To prove this, suppose \(\phi : \text{Spec } B \to Z\) is a \(G\)-invariant morphism. Then for each point \(P \in \text{Spec } B\), pick an affine neighborhood \(U = \text{Spec } C\) of \(Z\) containing \(f(P)\), and look at \(f^{-1}(U)\). Pick an element \(b_0\) which vanishes along the complement of \(f^{-1}(U)\), but does not vanish at any of the primes \(g \cdot P\) — algebraically, this corresponds to the fact that if \(I \subset B\) is an ideal which is not contained in any of the prime ideals \(Q_1, \ldots, Q_n\), and none of these ideals contain each other, then there is an element \(i \in I\) which is not in any of the \(Q_i\). Now set \(b = \prod_{g \in G} g \cdot b_0\). Then \(\text{Spec } B_b\) is an open subset containing \(P\) which maps to \(\text{Spec } C\) under \(f\). Thus \(f | \text{Spec } B_b\) corresponds to a map \(f^\# : C \to B_b\) which is \(G\)-invariant, and thus lands in \((B_b)^G = (B^G)_b\) by the next lemma. Thus, the map \(f | \text{Spec } B_b\) factors uniquely through \(\text{Spec } B^{G}\). Gluing, we see that \(f\) factors uniquely through \(\text{Spec } B^{G}\) and the claim follows.

\[\square\]

Lemma 2.7. For any ring \(B\) acted on by a finite group \(G\), and element \(b \in B^G\), we have that \(\phi : (B^G)_b \cong (B_b)^G\).

Proof. Since localization is flat, \((B^G)_b\) injects into \(B_b\), which means that \(\phi\) is injective.

To show surjectivity, suppose \(s/b^m \in (B_b)^G\). That means that as an element of \((B_b)\), \(s/b^m\) is invariant by \(G\), which means that for all \(g \in G, g \cdot s = s_g\) where \(s_g \cdot b^m = 0\) inside \(B\) for some \(n\). Taking \(n\) large enough for all \(g \in G\), we see that \(b^n \cdot s \in B^G\), and thus \(b^n \cdot s \in \text{im}\phi\), and thus so is \(s/b^m\).

We now focus on quotients of finite etale morphisms.

Theorem 2.8. Let \(f : Y \to X\) be Galois, and \(H\) be a subgroup of the automorphism group \(G = \text{Aut}_X Y\). Then \(H \setminus Y \to X\) is finite etale, and
\[G \setminus \pi_Y^{-1}(\overline{\tau}) \cong \pi_{G \setminus Y}^{-1}(\overline{\tau}).\]

Proof. Since finite maps are affine, we know from the lemma 2.6 that the quotient exists. Moreover, the statement is local on \(X\), so we may set \(X = \text{Spec} A\).
Spec $A,Y = \text{Spec } B, G \setminus Y = \text{Spec } B^G$. Now, pick any point $P \in \text{Spec } A$. Then $B \otimes_A k(P)$ is an etale extension of $k(P)$, hence is a direct sum of fields $\otimes_i L_i$ separable over $k(P)$. Moreover, the group $G$ acts simply transitively on the geometric points above $k(P) \to \overline{k(P)}$, and thus all the $L_i$ must be isomorphic, so that $B \otimes_A k(P) \cong L^r$ for some field Galois field $L$, the group $G$ acts simply transitively on the direct summands and, the stabilizer of each summand $L$ is isomorphic to its Galois group over $k(P)$.

Now, let $t$ be an element in the first component of $L$ such that its conjugates over $k(P)$ form a basis for $L$ over $k(P)$—that such an element exists is the normal basis theorem. Then it follows that $g \cdot t, g \in G$ form a basis for $B \otimes_A k(P)$ over $k(P)$. Now let $t$ be any lift of $t$ to $B_P$. Since we know $B_P$ is finite flat over $A_P$ it must be free, and so it follows that $g \cdot t, g \in G$, being the lift of a basis over $k(P)$ is a basis over $A_P$. Moreover, by spreading out as usual we can find $a \in A \setminus P$ such that $t$ lifts to $A_a$ an $g \cdot t, g \in G$ is a basis for $B_a$ over $A_a$. Since we are trying to prove a local statement, we may replace $A$ by $A_a$ and thus reduce to the case where $B$ is isomorphic as an $A[G]$ module to $A[G]$. It thus follows that $B^H \cong A[H \setminus G]$ and thus its finite flat over $A$.

To check that $B^H$ is etale over $A$ as well as the last assertion, it suffices to notice that for any $A$-algebra $C$, $(B \otimes_A C)^H = B^H \otimes_A C$. Thus, as we may check unramifiedness over geometric points, we may base change to $\text{Spec } \overline{k}$ via $\overline{\pi}$ and reduce to the case $A = \overline{k}$. But in this case, $B = \overline{k}^{[G]}$, so that $B^H = \overline{k}^{H \setminus G}$. The last assertion and the fact that $B^H$ is etale over $A$ are immediate consequences.

\[\square\]

**Corollary 2.9. For any connected finite etale $Z$ over $X$, and Galois $Y$ over $X$ with a surjection $\phi : Y \to Z$, there is a subgroup $H = \text{Aut}_Z Y < \text{Aut}_X Y$ such that $\phi$ induces an isomorphism from $H \setminus Y$ to $Z$. $Z$ is Galois over $X$ iff $H$ is normal in $\text{Aut}_X Y$, in which case $\text{Aut}_X Z = \text{Aut}_X Y / \text{Aut}_Z Y$.

**Proof.** Consider a geometric point $\overline{y}$ of $Y$ above $\overline{x}$ in $X$, and set $\overline{z} = \phi(\overline{y})$. Then as $\phi$ is of degree $[Y : Z] = \frac{[Y : X]}{[Z : X]}$, there must be $[Y : Z]$ geometric points in $Y$ above $\overline{x}$. Let us write these points as $(h_i \cdot \overline{y})_{i=1,...,n}$ where $h_i \in \text{Aut}_X Y$. Now, since maps from $Y$ to $Z$ are determined by the image of $\overline{y}$, it follows that $h_i$ are precisely the elements $g$ of $G$ such that $\phi = \phi \circ g$. Thus, $H = \{h_1, \ldots, h_n\}$ is the subgroup $\text{Aut}_Z Y$ of $\text{Aut}_X Y$, and so we get an induced map $H \setminus Y \to Z$. By the previous theorem, $H \setminus Y$ is etale over $X$, and
and thus it must also be etale over $Z$. But now

$$[H \setminus Y : Z] = \frac{[Y : H \setminus Y]}{[Y : Z]} = \frac{|H|}{|H|} = 1.$$ 

Since a finite flat map of degree 1 must be an isomorphism, the first part of the claim follows.

Now, since maps from $Y$ to $Z$ are determined by where $\overline{y}$ goes, and $\text{Aut}_X Y$ acts transitively on $\pi_Y^{-1}(\overline{y})$, it follows that $\text{Aut}_X Y \to \text{hom}_X(Y, Z)$ is surjective. Thus,

$$\text{Aut}_X Z = \text{hom}_X(Y, Z)^H = (H \setminus \text{Aut}_X Y)^H = \text{hom}_{\text{Aut}_X Y}(H \setminus \text{Aut}_X Y, H \setminus \text{Aut}_X Y).$$

Now, in general for finite groups $H < G$, $\text{hom}_G(H \setminus G, H \setminus G)$ is of size at most $|H|$ with equality iff $H$ is normal in $G$, in which case $\text{hom}_G(H \setminus G, H \setminus G) \cong G/H$. The claim follows. \qed

### 3 The Etale Fundamental Group

Suppose $Y \to Z$ is a morphism of Galois schemes over $X$. By corollary 2.9 there is a natural surjection $\text{Aut}_X Y \to \text{Aut}_X Z$. Now given a universal cover $(\tilde{X}, \tilde{x})$ we make the following definition:

**Definition.** We define $\pi_1(X, \overline{x})$ to be the profinite group $\varprojlim \text{Aut}_X(X_i)$, where the inverse limit is over all $i \in I$. One may consider $\pi_1(X, \overline{x})$ to be the automorphism group of $\tilde{X}$, in that it records all the compatible automorphisms of the $X_i$. In other words, $\pi_1(X, \overline{x})$ is that closed subgroup of $\prod_{i \in I} \text{Aut}_i X_i$ compatible with the transition maps.

As it stands, the group $\pi_1(X, \overline{x})$ depends on our choice of universal cover, but as we will see that’s not the case. However, before we give a more conceptual proof, the reader might want to try and prove it directly via the following exercise:

- Given two based universal covers $\varprojlim_{i \in I} (X_i, \overline{x}_i), \varprojlim_{j \in J} (X_j, \overline{x}_j)$ of $X$, prove that there is a natural isomorphism between $\varprojlim \text{Aut}_X X_i$ and $\varprojlim \text{Aut}_X X_j$ as topological groups. **Hint:** Construct a map by picking morphisms from the $X_i$ to the $X_j$, and use the base points to make the choice of morphism canonical.
3.1 Main Theorem: From covering spaces to $\pi_1(X, \overline{x})$-sets

Given a $Y \in \text{FET}/X$ we would like to construct a $\pi_1(X, \overline{x})$-set. By lemma 2.5 we have an isomorphism between $\pi^{-1}_Y(\overline{y})$ and $\text{hom}_X(\tilde{X}, Y)$. Moreover, we have a left action of $\pi_1(\tilde{X}, x)$ on $\text{hom}_X(\tilde{X}, Y)$ as follows: given $g \in \text{Aut}_X X_i$ and $\phi \in \text{hom}_X(X_i, Y)$ we define $g \cdot \phi := \phi \circ g^{-1}$. It is easy to check that this defines a \textit{continuous} action of $\pi_1(X, \overline{x})$ on $\text{hom}_X(\tilde{X}, Y)$, and thus on $\pi^{-1}_Y(\overline{y})$ via the above identification.

We write $F_Y$ for the finite-discrete $\pi_1(X, \overline{x})$ - module produced in this way

**Theorem 3.1.** The functor $Y \rightarrow F_Y$ defines an equivalence of categories:

$$F : \text{FET}/X \cong \{ \text{finite discrete } \pi_1(X, \overline{x}) - \text{sets} \}.$$ 

**Proof.** We first prove that $F$ is essentially surjective. Let $M$ be a finite discrete $\pi_1(X, \overline{x})$-set. Since $M$ is finite and discrete, the action on $M$ factors through $G := \text{Aut}_X X_i$ for some $i \in I$. Now write $M \cong \bigcup_{i=1}^n G_i/H_i$. Define $Y$ to be $\bigcup_{i=1}^n H_i \setminus X_i$. We claim that $F_Y$ is isomorphic to $M$.

Note that there are maps from $X_i$ to $H_i \setminus X_i$ sending $\overline{x}_i$ to any geometric point over $\overline{x}$, and thus it follows that $\text{hom}_X(\tilde{X}, H_i \setminus X_i) \cong \text{hom}_X(X_i, H_i \setminus X_i)$. Moreover, $\text{Aut}_X X_i/H_i \rightarrow \text{hom}_X(X_i, H_i \setminus X_i)$ is an injective map of sets with equal cardinality by lemma 2.8, and thus is a bijection. It follows that

$$\text{hom}_X(\tilde{X}, H_i \setminus X_i) = \text{Aut}_X X_i/H_i \cong G_i/H_i,$$

and one easily checks that the action of $G_i$ is via left multiplication, as desired.

It remains to prove that $F$ is fully faithful. So suppose that $Y, Z$ are finite etale over $X$. Pick $i \in I$ so that $X_i$ dominates both $Y$ and $Z$. By corollary 2.9 there are subgroups $H_Y, H_Z$ so that $Y \cong H_Y \setminus X_i$ and $Z \cong H_Z \setminus X_i$. We thus compute

$$\text{hom}_X(Y, Z) = \text{hom}_X(H_Y \setminus X_i, H_Z \setminus X_i)$$

$$= \text{hom}_X(X_i, H_Z \setminus X_i)^{H_Y}$$

$$= (H_Z \text{Aut}_X X_i)^{H_Y}$$

$$\cong \text{hom}_{G_i}(H_Y \setminus G_i, H_Z \setminus G_i)$$

$$\cong \text{hom}_{\pi_1(X, \overline{x})}(F_Y, F_Z)$$

As desired. 

\[\square\]
We leave it to the reader to prove the analogue of theorem 1.2 for profinite groups. This gives a definition of $\pi_1(X, x)$ independent of the choice of a universal cover.