MAT415: HOMEWORK SET #6

DUE: MONDAY APRIL 20, 2015

Define the finite Adelle ring $\mathbb{A}_f$ as follows: it is the subring of $\prod_p \mathbb{Q}_p$ (where the product is over all primes p) consisting of those elements

$$\vec{a} = (a_2, a_3, a_5, \dots, a_p, \dots)$$

such that for almost all prime p, $a_p \in \mathbb{Z}_p$.

(1) Prove that the diagonal embedding $\mathbb{Q} \hookrightarrow \prod_p \mathbb{Q}_p$ gives a map $\mathbb{Q} \to \mathbb{A}_f$. So that $\mathbb{A}_f$ naturally contains $\mathbb{Q}$ as a subring.

(2) Define a metric on $\mathbb{A}$ by $|\vec{\alpha}|_{\mathbb{A}_f} := \max_p p^{-\frac{1}{2}} |\alpha_p|_p$. This makes $\mathbb{A}_f$ into a metric space by $d(\vec{\alpha}, \vec{\beta}) := |\vec{\alpha} - \vec{\beta}|_{\mathbb{A}_f}$. Prove that $\mathbb{Q}$ is dense in $\mathbb{A}_f$.

(3) In the same way as above, one can consider the diagonal embedding of $\mathbb{Q}$ into $\mathbb{A}_f \times \mathbb{R}$. Prove that the image of $\mathbb{Q}$ in $\mathbb{A}_f \times \mathbb{R}$ is discrete.

(4) For an abelian group G, its pontryagin dual is defined to be $\widehat{G} := \mathrm{Hom}_{cont}(G, \mathbb{R}/\mathbb{Z})$, the group of continuous homomorphisms of from G to $\mathbb{R}/\mathbb{Z}$. In what follows, consider $\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{A}_f$ as abelian groups under addition. Prove that
   (a) $\widehat{\mathbb{Z}} \approx \mathbb{R}/\mathbb{Z}$
   (b) $\widehat{\mathbb{Q}_p/\mathbb{Z}_p} \approx \mathbb{Z}_p$
   (c) $\widehat{\mathbb{Q}/\mathbb{Z}} \approx \prod_p \mathbb{Z}_p$
   (d) BONUS: $\widehat{\mathbb{Q}} \approx (\mathbb{A}_f \times \mathbb{R})/\mathbb{Q}$.