MAT415: HOMEWORK SET #2

DUE: MONDAY FEBRUARY 23, 2015

(1) Determine the class group of $\mathbb{Q}(\sqrt{-23})$.
(2) Determine the class group of $\mathbb{Q}(\sqrt{-111})$.
(3) Let K be a number field of degree n over $\mathbb{Q}$. For any lattice $L \subset K$, define the discriminant Disc_L s follows: Let $\alpha_1, \ldots, \alpha_n$ be a $\mathbb{Z}$-basis for L. Consider the $n \times n$ matrix M whose entries are

$$M_{ij} = \mathrm{tr}_{K/\mathbb{Q}}(\alpha_i \alpha_j).$$

Then Disc_L is defined to be the determinant of M.
   (a) Prove that the definition of Disc_L is independent of the choice of basis.
   (b) Prove that if L' is another lattice containing L, then $\mathrm{Disc}_L = \mathrm{Disc}_{L'} \cdot |L'/L|^2$.
   (c) Prove that the discriminant D_K of K is equal $\mathrm{Disc}_{\mathcal{O}_K}$.
(4) Let $K = \mathbb{Q}(\sqrt[3]{2}) \cong \mathbb{Q}[x]/(x^3 - 2)$.
   (a) Prove that K is not Galois over $\mathbb{Q}$.
   (b) Let L be the lattice of K spanned (as a $\mathbb{Z}$-module) by $1, x, x^2$. Prove that L is a subring of $\mathcal{O}_K$.
   (c) Compute Disc_L, and prove that either $L = \mathcal{O}_K$ or that the index of L in $\mathcal{O}_K$ is 2. **Hint: Use the Minkowski bound.**
   (d) Prove that $L = \mathcal{O}_K$.
   (e) Prove that the class number of K is 1.
   (f) Determine the unit group of K.
(5) Determine the unit group and class group of $\mathbb{Q}(\sqrt{145})$.
(6) Determine the unit group and class group of $\mathbb{Q}(\sqrt[3]{7})$.
(7) Let $K \subset \mathbb{C}$ be a number field, and suppose that K is normal over $\mathbb{Q}$.
   (a) Prove that K is of degree 1 or 2 over $K \cap \mathbb{R}$.
   (b) Prove that $F = K \cap \mathbb{R}$ is normal over $\mathbb{Q}$ iff F has no non-real embeddings
   (c) Let $U_K = \mathcal{O}_K^\times$ be the group of units. Prove that $U_K \cap \mathbb{R}$ is finite index in U_K iff the automorphism of K induced by complex conjugation commutes with all automorphisms of K.
(8) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$. Let $U_K = \mathcal{O}_K^\times$ be the group of units of K. Find generators for U_K as a finite abelian group, and compute the relations between them. **Hint: Use the action of the Galois group on U_K to (almost) reduce the problem to computing the group of units the 3 quadratic subfields of K.**