MAT415: HOMEWORK SET #1

DUE: MONDAY FEBRUARY 2, 2015

(1) Show that $\frac{\sqrt{3}+\sqrt{7}}{2}$ is an algebraic integer, but $\frac{\sqrt{3}+\sqrt{5}}{2}$ is not.

(2) Prove that the rings of integers of $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{-6})$ are $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ and $\mathbb{Z}[\sqrt{-6}]$ respectively.

(3) Find an integral basis for the ring of integers in $\mathbb{Q}(\sqrt{3}, \sqrt{2})$.

(4) Let I be the ideal $(2, 1+\sqrt{-3})$ in $\mathbb{Z}[\sqrt{-3}]$. Prove that $I^2 = 2I$ but $I \neq (2)$. Conclude that ideals in $\mathbb{Z}[\sqrt{-3}]$ do not factor uniquely into prime ideals, and thus $\mathbb{Z}[\sqrt{-3}]$ is not the ring of algebraic integers in $\mathbb{Q}(\sqrt{-3})$.

(5) By considering the factorization of 6, prove that $\mathbb{Z}[\sqrt{-6}]$ is not a unique factorization domain, and factor (6) into prime ideals.

(6) Let $K = \mathbb{Q}(\sqrt[3]{2})$. Prove (using the trace pairing or otherwise) that the ring of integers of K is $\mathbb{Z}[\sqrt[3]{2}]$.

(7) You may assume that the ring of integers in $\mathbb{Q}[\sqrt{-14}]$ is $\mathbb{Z}[\sqrt{-14}]$. Let $I = (3, \sqrt{-14} - 1)$ be an ideal in $\mathbb{Z}[\sqrt{-14}]$. Prove that I, I^2, I^3 are not principal, while I^4 is.

(8) Let R be a domain, and K be its fraction field. We say that $x \in K$ is *integral* over R if the ring $R[x]$ is finite as an R-module. Let S denote the set of all elements $x \in K$ which are integral over R. Prove that S is a ring. We say that S is the *integral closure* of R, and that R is *integrally closed* if $R = S$.

(9) We say that a domain R is *of dimension one* if every prime ideal in R is maximal.

Let R be a domain with fraction field K, and define a *fractional ideal* of K to be any R-submodule $M \subset K$ such that there exists $r \in R$ with $r \neq 0$ such that $rM \subset R$. Let I_K denote the set of non-zero fractional ideals. Prove that the following are equivalent:

(a) R is Noetherian, integrally closed, and of dimension one.

(b) I_K forms a group under multiplication.

We say that a ring R satisfying the above is a *Dedekind Domain*.