MAT415: TAKE HOME FINAL EXAM

DUE: FRIDAY MAY 1, 2015

Instructions: Please work on this exam by yourself, without consulting your peers or anyone else. You may use books and notes as references, but please do not attempt to find the same exact question online.

(1) (a) (5 marks) Prove that a prime number p can be written as

$$x^2 + xy + 2y^2$$

for integers x, y iff p is either $0, 1, 2$ or 4 modulo 7.

(b) (5 marks) Prove that a prime number $p \neq 5$ can be written as

$$x^2 + 5y^2$$

for integers x, y iff p is either 1 or 9 modulo 20.

(2) (10 marks) We say that a finite extension of number fields L/K is *unramified* if for every prime ideal P of K and prime ideal Q of L with Q dividing $P\mathcal{O}_L$, the ramification index $e(Q|P)$ is equal to 1. Prove that $K = \mathbb{Q}(\sqrt{21})$ has a unique quadratic extension L which is unramified over K, and find L (you may present L as either $\mathbb{Q}[x]/(f(x))$ or $K[x]/(f(x))$.)

(3) Let K be a number field, with $[K : \mathbb{Q}] = 3$ and such that K is Galois over $\mathbb{Q}$. Let r be the numbers of primes dividing the discriminant of K (i.e. the number of primes that ramify in K). Let $CL(K)[3]$ denote the 3-torsion subgroup of the class group of K, and let $|CL(K)[3]| = 3^s$.

(a) (5 marks) Prove that $s \geq r - 1$.
(b) (5 marks) Prove that $s \leq 2r$.
(c) (2 marks Bonus:) Prove that $s \leq 2r - 2$.

Hint: For this problem, you may find it useful to use Hilbert's Theorem 90: If σ is a generator for Gal$(K/\mathbb{Q})$, and $x \in K^\times$ is such that $Norm_{K/\mathbb{Q}}(x) = x\sigma(x)\sigma^2(x) = 1$ then there exists $y \in K^\times$ such that $x = \sigma(y)/y$. You may quote this theorem without proof.

(4) Let p and q be two distinct prime numbers.

(a) (5 marks) Prove that $\mathbb{Q}_p$ is not isomorphic to $\mathbb{Q}_q$.
(b) (5 marks) Prove that $\mathbb{Q}_p$ has no field automorphisms other than the identity. **Warning: Field automorphisms do NOT have to be continuous...**.

(5) (10 marks) Let $\mathbb{Q}_7$ be the field of 7-adic integers. Prove that there does not exist a Galois extension L of $\mathbb{Q}_7$ with Galois Group isomorphic S_3, the symmetric group on 3 elements.