AX-SCHANUEL AND O-MINIMALITY

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1. Interpreting Ax-Schanuel Geometrically

The goal of this note is to give a geometric interpretation of the Ax-Schanuel theorem, and to give a model-theoretical proof of it. To start, lets recall the

Theorem 1.1 (Ax-Schanuel). *Let $f_1, \dots, f_n \in \mathbb{C}[[t_1, \dots, t_m]]$ be power series that are $\mathbb{Q}$ - linearly independent modulo $\mathbb{C}$. Then we have the following inequality:*

$$\dim_{\mathbb{C}} \mathbb{C}(f_1, \dots, f_n, e(f_1), \dots, e(f_n)) \geq n + rank \left(\frac{\partial f_i}{\partial t_j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$$

where $e(x) = e^{2\pi i x}$ and $\dim_K L$ is the transcendence degree of L over K.

To see the geometric implication of this theorem, lets restrict to the case where the power series f_i are convergent in some open neighborhood $B \subset \mathbb{C}^m$. Note that by the Seidenberg embedding theorem[1], it is sufficient to look at this case. Define the uniformizing map

$$\pi_n : \mathbb{C}^n \to (\mathbb{C}^\times)^n, \pi_n(x_1, \dots, x_n) = (e(x_1), \dots, e(x_n))$$

and the subset $D_n \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ to be the set

$$(\vec{x}, \vec{y}) \in D_n \iff \pi_n(\vec{x}) = \vec{y}.$$

Then we have a well defined map $\vec{f} : B \to D_n$ given by

$$\vec{f}(t_1, \dots, t_m) = (f_1(t_1, \dots, t_m), \dots, f_n(t_1, \dots, t_m), e(f_1(t_1, \dots, t_m)), \dots, e(f_n(t_1, \dots, t_m)).$$

Define $U \subset D_n$ to be the image of $\vec{f}$. U is then a complex analytic space, and it is easy to verify that

$$\dim_{\mathbb{C}}(U) = \text{rank} \left(\frac{\partial f_i}{\partial t_j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$$

and denoting the Zariski closure of U by U^{zar},

$$\dim_{\mathbb{C}}(U^{zar}) = \dim_{\mathbb{C}} \mathbb{C}(f_1, \dots, f_n, e(f_1), \dots, e(f_n)).$$

[1]Thanks to Martin Bays for pointing this out

Moreover, denote by π_a and π_m the projections onto $\mathbb{C}^n$ and $(\mathbb{C}^\times)^n$ respectively[2]. Then the linear independence condition on the f_i is equivalent to saying that $\pi_a(U)$ does not lie in the translate of a proper $\mathbb{Q}$-linear subspace of $\mathbb{C}^n$, or that $\pi_m(U)$ does not lie in a coset of a proper subtorus. We can thus rephrase the Ax-Schanuel theorem geometrically as follows:

Theorem 1.2 (Ax-Schanuel, V.2). *Defining D, π_m as above, let $U \subset D_n$ be an (irreducible) complex analytic subspace such that $\pi_m(U)$ does not lie in a coset of a proper subtorus of $(\mathbb{C}^\times)^n$. Then*

$$\dim_{\mathbb{C}} U^{zar} \geq \dim_{\mathbb{C}} U + n$$

where U^{zar} denotes the Zariski closure of U in $\mathbb{C}^n \times (\mathbb{C}^\times)^n$.

It is more convenient to rephrase the above as a theorem about subvarieties of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$. This is easy to do by starting with U^{zar} instead of U. The following rephrasing is then equivalent to the above:

Theorem 1.3 (Ax-Schanuel, V.3). *Let $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an irreducible subvariety, and let U be a connected, irreducible component of $V \cap D_n$. Assume that $\pi_m(U)$ is not contained in a coset of a proper subtorus of $(\mathbb{C}^\times)^n$. Then*

$$\dim_{\mathbb{C}} V \geq \dim_{\mathbb{C}} U + n.$$

Remark. *It is easy to see that the above version implies the Ax-Lindemann-Weierstrass theorem: Suppose that $V_1 \subset \mathbb{C}^n$ and $V_2 \subset (\mathbb{C}^\times)^n$ are irreducible varieties with $V_1 \subset \pi^{-1}(V_2)$. Then plugging in $V = V_1 \times V_2$ into theorem 1.3, we see that $V \cap D$ has dimension at least as high as V_1. The theorem then implies that $\dim(V_2)$ is at least n and hence that V_2 must be all of $(\mathbb{C}^\times)^n$.*

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2. An o-minimality proof of Ax-Schanuel

This entire section is devoted to a proof of theorem 1.3 using the techniques of Pila-Zannier. To start with, we can assume that $U^{zar} = V$. We proceed by induction, the induction being lexicographic on the triple $(n, \dim V - \dim U, n - \dim U)$. The case of U being a point is trivial, we assume U has positive dimension. By convention, definable always means definable in $\mathbb{R}_{\text{an,exp}}$.

Definition. *For an irreducible analytic set $X \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$, we define X^{Lin} to be smallest affine linear subvariety containing $\pi_a(X)$.*

[2]The reason for the notation is that the additive and multiplicative groups are denote $\mathbf{G}_a$ and $\mathbf{G}_m$.

Define

$$F = \{(z_1, \ldots, z_n, w_1, \ldots, w_n) \in \mathbb{C}^n \times (\mathbb{C}^\times)^n \mid 0 \leq Re(z_i) \leq 1\}$$

and note that $D_n \cap F$ is definable. Then $U \cap F$ is definable. Moreover, for an analytic set $X \subset \mathbb{C}^\times \times (\mathbb{C}^\times)^n$ and a linear subspace $L \subset \mathbb{C}^n$ we define $G_d(X, L)$ to be the set of points $x \in X$ around which X is regular of dimension d, and such that the irreducible component X_0 containing x satisfies X_0^{Lin} is a translate of L.

Let $I \subset \mathbb{R}^n$ be defined by

$$I = \{\ell \in \mathbb{R}^n \mid G_{\dim U}\left(\left((\ell + V)\bigcap(D_n \cap F)\right), U^{\mathrm{Lin}}\right) \neq \emptyset\}$$

where addition is defined by acting on the first n co-ordinates of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$. Then I is definable, and we're going to get somewhere by considering the intersection of I with $\mathbb{Z}^n$, the monodromy group.

Define $F_{\vec{m}} = F + \vec{m}$ and note that $\bigcup_{\vec{m} \in \mathbb{Z}^n} F_{\vec{m}} = \mathbb{C}^n \times (\mathbb{C}^\times)^n$. Moreover, if $U \cap F_{\vec{m}} \neq \emptyset$ then $-\vec{m} \in I$. This is because

$$(U \cap F_{\vec{m}}) - \vec{m} = (U - \vec{m}) \cap F \subset (V - \vec{m}) \cap D_n \cap F$$

where we have used the fact that $D_n + \vec{m} = D_n$. Assume first that $I \cap \mathbb{Z}^n$ is finite. In this case, it follows that U is a finite union of $U \cap F_{\vec{m}}$ and so is definable. Hence U is definable, closed and analytic in $\mathbb{C}^n \times (\mathbb{C}^\times)^n$, and so by [1, Theorems 4.5 and 5.3], U must be an algebraic variety. However, it is trivial to show that D_n contains no positive dimensional algebraic varieties (f and $e(f)$ can't both be algebraic functions for growth reasons, for example) which is a contradiction.

We thus conclude that $I \cap \mathbb{Z}^n$ is infinite. In particular, U intersects infinitely many $F_{\vec{m}}$. However, since U is connected the set of vectors $\vec{m}$ such that $U \cap F_{\vec{m}} \neq \emptyset$ must be a connected set in the graph G with vertex set $\mathbb{Z}^n$ and where the edges are given by connecting pairs of vertices all of whose co-ordinates are off by at most 1. But now we get for free that $I \cap \mathbb{Z}^n$ has at least T integer points of height at most T. Applying the counting theorem of Pila-Wilkie ([2],Thm 1.9) we conclude that I contains a semi-algebraic curve $C_{\mathbb{R}}$, containing at least 1 smooth non-zero integer point $l \in C(\mathbb{Z})$. We refer to C as the corresponding complex algebraic curve.

Next, consider the algebraic variety $V + C$. For each $c \in C_{\mathbb{R}}$ consider an irreducible component W_c of $(V + C)\bigcap(D_n \cap F)$ of dimension $\dim U$, such that $W_c^{\mathrm{Lin}} = U^{\mathrm{Lin}}$. If there are infinitely many such components as c varies, then there must be a component W of $(V + C) \cap D_n$ containing infinitely many such W_c. Hence W is of dimension at least $\dim U + 1$. Moreover, since $\pi_a(U)$ is not contained in a coset of a $\mathbb{Q}$-linear space it implies that U^{Lin} isn't and hence $\pi_a(W)$ isn't either. Thus we can replace V and U by $V + C$ and W and induct.

Otherwise, there must be only finitely many such W_c. Hence, there must be such a component $W = W_c$ appearing in infinitely many translates $V + c$, and thus by analyticity in all such translates by $c \in C$. If V is not invariant by translation under all elements of C, we replace (U, V) by $(W, \bigcap_{c\in C} V + c)$ and induct. Thus, we may assume that $V + C = V$.

In particular, V is invariant under l, hence also under the complex line generated by l by algebraicity. We make a linear change of co-ordinates with $\mathbb{Z}$-coefficients in $\mathbb{C}^n$ so that l is a multiple of $(1, 0, \dots, 0)$ and the corresponding 'monomial' change of coordinates in $(\mathbb{C}^\times)^n$ so as to keep D_n invariant - note that this change of co-ordinates preserves all relevant dimension. We can thus write V as $V = \mathbb{C} \times V^0$ where

$$V^0 \subset \mathbb{C}^{n-1} \times (\mathbb{C}^\times)^{n-1} \times \mathbb{C}^\times.$$

The idea now is to apply induction on n.

So write $D_n = D_1 \times D_{n-1}$ and $U = \bigcup_{z\in D_1} z \times U_z$. For $z \in D_1$, let $V_z \subset \mathbb{C}^{n-1} \times (\mathbb{C}^\times)^{n-1}$ denote the fiber of V over z. Note that since U surjects under projection onto an open set of D_1, so must V and since V is algebraic it must be dominant onto $D_1^{zar} = \mathbb{C} \times \mathbb{C}^\times$. Thus, $\dim V = 2 + \dim V_z$ for a generic z. Now, we split into two cases:

- Suppose that the $\pi_a(U_z) \subset \mathbb{C}^{n-1}$ are not generically contained in a proper $\mathbb{Q}$-linear subspace. Then by induction, we have that for a generic z

  $$\dim V_z \geq n - 1 + \dim U_z$$

  which yields our claim since $\dim V = \dim V_z + 2$ while $\dim U = \dim U_z + 1$.
- Else, since U is not contained in a proper $\mathbb{Q}$-linear subspace the U_z must vary with z. Let $U_0 \subset D_{n-1}$ be the projection of U and $V_0 \subset \mathbb{C}^{n-1} \times (\mathbb{C}^\times)^{n-1}$ be the projection of V. Note that since the U_z vary, we have that $\dim U = \dim U_0$. Then by induction, $\dim V_0 \geq \dim U_0 + (n-1)$. This again yields our claim, since $\dim V \geq 1 + \dim V_0$.

References

[1] Y. Peterzil and S. Starchenko, Tame complex analysis and o-minimality, *Proceedings of the ICM, Hyderabad 2010*

[2] J. Pila and A. J. Wilkie, The rational points of a definable set, *Duke Math. J.* **133** (2006), 591–616.