# MAT137 - Term 2, Week 12

- This is our last lecture!
- Your tenth and final problem set is due next week, Thursday 5 April, at 11:59pm.
- Please fill out your course evaluations.
  - They really do matter!
  - Please write some comments, if you have time.
  - You can do it any time between now and 8 April, through Portal, on the "Course Evals" tab.
- Today we will:
  - Talk about applications of Taylor series.
- I've included some exercises we didn't get a chance to talk about in class. They're indicated with red text.

# Taylor series

Last class we defined the Taylor series of a function $f$:

## Proposition

*Suppose $f$ is a $C^\infty$ function, and $a$ is a real number.*

*Then the Taylor series of $f$ at $a$ is the following power series:*

$$S(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

# Analyticity

Having defined the Taylor series of a function, we saw this function:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We learned that this function is $C^\infty$, but that its Taylor series is *identically zero*.

In order words, this function does not equal its Taylor series anywhere (except at 0).

A $C^\infty$ function is, roughly speaking, called <u>analytic</u> if it equals its Taylor series.

# Analyticity

Proving that a function is analytic is usually tricky. We learned about "remainder theorems" that let us estimate the error when we approximate a function with a Taylor polynomial.

In one of the videos you saw a proof that $e^x$ is analytic, and last class you used Lagrange's Remainder Theorem to prove that $\sin(x)$ is analytic.

Here, for your future reference, is the full definition:

**Definition**

Let $f$ be a function defined at least on an open interval $I$.

$f$ is <u>analytic on $I$</u> if for every $a \in I$, the Taylor series of $f$ centred at $a$ converges to $f(x)$ for all $x$ near $a$.

# Other analytic functions

So far you know (or may now assume that) these four functions are analytic:

$$e^x \qquad \sin(x) \qquad \cos(x) \qquad \frac{1}{1-x}.$$

(The first three being analytic on $\mathbb{R}$, and the last being analytic on $(-1,1)$.) All polynomials are also analytic, of course, since they *are* their own Taylor series.

You also know that sums, products, compositions, derivatives, and antiderivatives of analytic functions are analytic.

Last class you used the facts above to prove that:

- $e^{-x^2}$ is analytic on $\mathbb{R}$,
- $\log(1+x)$ is analytic on $(-1,1)$,
- $\arctan(x)$ is analytic on $(-1,1)$.

# The six most importan Taylor series

Here are the six most important Taylor series we know so far.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad (R = \infty) \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (R = 1)$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad (R = \infty) \qquad \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad (R = 1)$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad (R = \infty) \qquad \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \qquad (R = 1)$$

The last two can be derived quite easily from the fourth one though.

# Exercises from last class

**Exercise:** Compute the Taylor series of the following functions, at the specified centre points.

1. $x^5 \log(1 + x^3)$, about 0.
2. $e^x(1 - x^2)$ about 0.
3. $\cos^2 x$, about 0. (Hint: use a trig identity.)
4. $\sin(\pi x)$, about $\frac{1}{2}$. (Don't overthink this one.)
5. $\dfrac{1}{x}$, about a fixed, nonzero real number $a$.

**Exercise:** For each part of the previous question:

Call the function in question $f$, and let $a$ be the point at which the series is centred. Find the value of $f^{(100)}(a)$.

# Using Taylor series to evaluate limits

Taylor series are very, very useful for evaluating limits.

This is the thing your instructors and TAs usually do in their head when given a tricky limit.

**Exercise:** Compute the following limit:

$$\lim_{x \to 0} \frac{\cos(x) - 1 + \frac{1}{2}\,x^2 e^x}{x^3}.$$

This can be done with L'Hôpital's rule, but it's much easier with Taylor series. Just find the first non-zero term in the Taylor series of the numerator and denominator.

# More exercises

**Exercise:** Compute the following limit:

$$\lim_{x \to 0} \frac{\cos(x) - 1 + \frac{1}{2} x \sin(x)}{(\log(1+x))^4}.$$

**Exercise:** Compute the following limit:

$$\lim_{x \to 0} \frac{\left(x\, e^x + \sin(3x) - x^2\right)(\cos(x) - 1)}{2 \sin(x^2)(e^{\pi x} - 1)}$$

**Exercise:** Compute the following limit:

$$\lim_{x \to 0} \frac{\log(\cos(x))}{\sin(7x^2)}$$

# Using Taylor series to evaluate series

What function is this the Taylor series of?

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+2}$$

We notice that $\frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+2}$ is an antiderivative of

$$\frac{(-1)^n}{(2n+1)} x^{2n+1}.$$

These are the terms in the Taylor series of $\arctan(x)$. So our original series is the Taylor series of

$$\int \arctan(x)\, dx = x \arctan(x) - \frac{1}{2} \log(x^2+1).$$

(The constant of integration here is 0.)

# Using Taylor series to evaluate series

Now what about the following series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+7}$$

If we factor out $x^5$, we get the previous series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+7} = x^5 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+2}$$

So this is the Taylor series of the function:

$$x^5 \int \arctan(x)\, dx = x^5 \left( \arctan(x) - \frac{1}{2} \log(x^2+1) \right)$$

# Example

These are the sorts of manipulation we use to compute the sums of series using Taylor series.

**Exercise:** Compute the value of the sum: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}(2n+1)(2n+2)}$.

Hint 1: Look at the previous problems!

Hint 2:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}(2n+1)(2n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} \left(\frac{1}{\sqrt{3}}\right)^{2n+2}.$$

Numerical final answer: $\displaystyle\frac{\pi}{6\sqrt{3}} - \frac{1}{2}\log\left(\frac{4}{3}\right)$

# More exercises

**Exercise:** Compute: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^{n+1}}$.

**Exercise:** Compute $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\pi^{2n}}{4^n\,(2n+1)!}$.

**Exercise:** Compute $\displaystyle\sum_{n=1}^{\infty} \frac{n^2\,7^n}{n!}$.

# Using Taylor series to estimate things

We didn't see this slide in class.

Last class you used Lagrange's Remainder theorem to prove that $\sin(x)$ is analytic on $\mathbb{R}$. Here's the theorem again:

**Theorem**

*Let $f$ be $C^{n+1}$ on an interval $I$ containing a point $a$.*

*Then for any $x \in I$, we have:*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

*for some number $c$ in between $a$ and $x$.*

**Exercise:** Use Lagrange's Remiander theorem to estimate the value of $\sin(7)$ to within an error of 0.001.

# Using Taylor series to estimate $\pi$

The number that has famously been estimated the most is $\pi$.

To use Taylor series to estimate $\pi$, we would like a function...

- that is analytic and whose Taylor series we know (or can easily compute), and
- that outputs $\pi$, or maybe a constant multiple of $\pi$.

Can you think of such a function?

$f(x) = \arctan(x)$ is the function for the job! We know its Taylor series, and we know that $\arctan(1) = \frac{\pi}{4}$.

# Using Taylor series to estimate $\pi$

**Exercise:**

1. Write down a series whose sum equals $\frac{\pi}{4}$.
2. Note that the series you wrote down is alternating.

   Now use the alternating series estimation technique you learned to figure out how many terms of the series you need to approximate $\pi$ to an error of less than 0.001.

   (ie. How many terms do you need to get between 3.141 and 3.143.)

This series converges *very* slowly. To get $\pi$ accurate to 10 decimal places, you need over five *billion* terms.

# Using Taylor series to estimate $\pi$

We have done much better over the years.

In 1706, John Machin proved this boring-looking formula:

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

Luckily, he later met Brook Taylor, who told him about his fancy new series.
If you write out the right side of this equation as a series, it converges *much* faster than the one you found earlier.

For example, after 11 terms of the series, the partial sum agrees with $\pi$ up to 15 decimal places.

# Approximating $\pi$ with series

Machin's formula and several others like it were the primary tools for approximating $\pi$ for a long time.

In 1910, the now legendary Indian mathematician Srinivasa Ramanujan proved this (somehow):

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103 - 26390n)}{(n!)^4\, 369^{4n}}.$$

This series yields eight correct decimal digits of $\pi$ *per term*.

This series is the basis for the current fastest algorithms for appoximating $\pi$.

# Approximating $\pi$ with series

As far as I know, this is the best one of these series currently known:

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n \, (6n)! \, (13591409 + 545140134n)}{(3n)! \, (n!)^3 \, (640320)^{3n+3/2}}.$$

This series yields a little more than 14 correct digits of $\pi$ per term, on average.

We currently know a bit more than the first 22 trillion digits of $\pi$.

# More applications.

We didn't see this slide in class.

**Exercise:** Use Taylor series to help prove whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n}\right)$$

This can also be answered with the integral test if you don't want to think about Taylor series, but in mathematics it is your duty to find the laziest way of doing everything. Taylor series are easier, so use those.