- Test 4 is **tomorrow**. Hopefully you already knew that! See the course website for details.
- Today we will:
  - Talk about Taylor polynomials and Taylor series.
- For next week (last lecture!):
  - Watch all remaining videos on playlist 14.
- You have a homework assignment from this lecture. See the last slide.

Towards Taylor series

Now we've convinced ourselves that power series are great, because the functions they define act just like polynomials so long as we're careful to work inside their radii of convergence.

So what? How does this help us do anything?

The next idea is to develop a framework for expressing functions we already know about as power series. That's what the machinery of Taylor series is about.

We come up with Taylor series by trying to approximate known functions more and more accurately with [normal, finite-degree] polynomials, and then "taking a limit". These polynomials are called Taylor polynomials.

Taylor polynomials

First we have to formalize what it means to approximate a function "better and better". This is a two part process.

Definition

Let f and g be two functions such that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ for some real number a.

We say <u>g approaches zero faster than f as $x \to a$</u> if

$$\lim_{x\to a} \frac{g(x)}{f(x)} = 0.$$

This definition gives us a way to compare two functions, and see which one is going to zero faster than the other.

Taylor polynomials

Now we define something like "f is a good approximation for g", but more quantitatively. Essentially, we want to "grade" or "measure" how good the approximation is.

Definition

Let f and g be two functions such that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ for some real number a.

Let n be a non-negative integer. We say
<u>g is a good approximation of order n for f near a</u> if

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0.$$

Idea: The bigger n is, the more closely g approximates f (near a).

Taylor polynomials

Finally, we can define a Taylor polynomial as the "least complex" polynomial that does this sort of approximation:

Definition

Let f be a continuous function defined at and near a real number a, and n be a non-negative integer.

The <u>n^{th} Taylor polynomial of f at a</u> is a polynomial P_n of smallest degree such that P_n is a good approximation of order n for f near a

Note that this definition doesn't require that f is differentiable any number of times. Just continuous!

Taylor polynomials

Some simple manipulations and applications of L'Hôpital's rule brought us to this result, which is a more concrete characterization of Taylor polynomials:

Definition

Let f be a C^n function at a real number a, and let n be a positive integer.

The n^{th} Taylor polynomial of f at a is a polynomial P_n of smallest degree such that

$$P_n(a) = f(a), P_n'(a) = f'(a), \ldots, P_n^{(n-1)}(a) = f^{(n-1)}(a), P_n^{(n)}(a) = f^{(n)}(a).$$

Note that the definition does not require that P_n is a degree n polynomial. Its degree can be less.

Exercise: What is the 3rd Taylor polynomial of $f(x) = \sin(x)$ at $a = 0$? What about the 7th? The 8th?

Exercises

Exericse: Let $f(x) = e^x$. Which of the following two functions is a better approximation of f near $a = 0$?

- $g_1(x) = 1 + x + x^2$.
- $g_2(x) = \sin(x) + \cos(x) + x^2 + \frac{1}{3}x^3$.
- $g_3(x) = e^{-x} + 2x$.

Exericse: Let f be a function that has all of its derivatives.

- Find an explicit formula for the 3^{rd} Taylor polynomial of f at 0.
- Let n be a non-negative integer. Find an explicit formula for the n^{th} Taylor polynomial of f at 0.

An explicit formula for Taylor polynomails

It's a routine exercise to prove that this formula is true (by induction). That gives us our third characterization of Taylor polynomials:

Theorem

Let n be a non-negative integer, let a be a real number, and let f be a C^n function defined near a.

The n^{th} Taylor polynomial of f at a has the following explicit form:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Taylor series

Now we are ready to define Taylor series, which are essentially "infinite degree" Taylor polynomials.

Definition

Suppose f is a function that has all of its derivatives, and a is a real number.

The Taylor series of f and a is the power series S, centred at a, such that

$$S^{(k)}(a) = f^{(k)}(a) \qquad \text{for all } k = 0, 1, 2, \ldots$$

In other words, the Taylor series of f at a is the power series S such that all the derivatives of S and f agree at a.

Taylor series

It is a simple exercise in induction to prove that the terms in the Taylor series of f at a look just like the terms in the Taylor polynomials of f at a:

Proposition

Suppose f is a function that has all of its derivatives, and a is a real number.

Then the Taylor series of f at a is the following power series:

$$S(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor series we know so far

You saw four important Taylor series in the videos:

$$e^x: \quad \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad = \quad 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \qquad (R = \infty)$$

$$\cos(x): \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad = \quad 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \qquad (R = \infty)$$

$$\sin(x): \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad = \quad x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \qquad (R = \infty)$$

$$\frac{1}{1-x}: \quad \sum_{n=0}^{\infty} x^n \quad = \quad 1 + x + x^2 + x^3 + \cdots \qquad (R = 1)$$

There's more going on here than you think.

CAUTION: We've never claimed that a function f *equals* its Taylor series!

For example, we have the Taylor series of $\cos(x)$ at $a = 0$:

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

but we've never claimed that $S(x) = \cos(x)$.

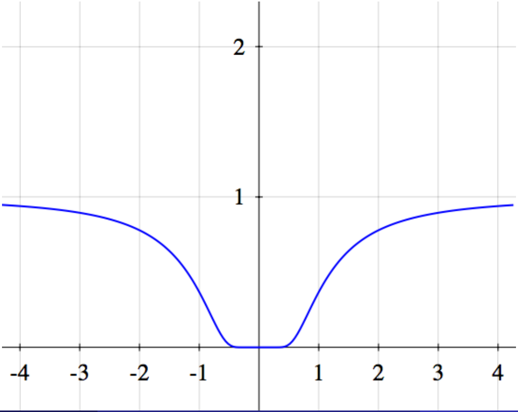
This is a very subtle and important point. First we'll give an example of how this can go wrong, then recall from the videos how we ensure it doesn't go wrong.

A very pathalogical example

Example: Consider the following function:

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here's the relevant part of its graph:



A very pathological example

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It is a a tedious (but purely computational) exercise to check that:

$$f^{(k)}(0) = 0 \quad \text{for all } k = 0, 1, 2, \ldots$$

You may assume this without proof.

Exercise:

- What is the Taylor series of this function at 0?
- Where does this series converge absolutely?
- For which points x does the function equal its Taylor series?

Now what?

The whole idea here was to use the Taylor series of a function to understand the function better, or manipulate it more easily.

So the next goal is to find out when and how we can guarantee a function f equals its Taylor series.

Such a function will be called an analytic function, roughly speaking.

For now just note that we just saw a function that has all of its derivatives everywhere, and whose Taylor series converges absolutely everywhere, but which is not analytic.

Minimizing error

Think back to our discussion of "good approximations of order n", error functions, etc.

Provided f has all of its derivatives at a, we can define

- P_n, its n^{th} Taylor polynomial at a, for all n, and
- S, its Taylor series at a.

We know that P_n is a good approximation of order n for f at a, and that this approximation likely has some error for values of x other than a. Call this error $R_n(x)$. In other words, we'll have that for all x, $f(x) = P_n(x) + R_n(x)$.

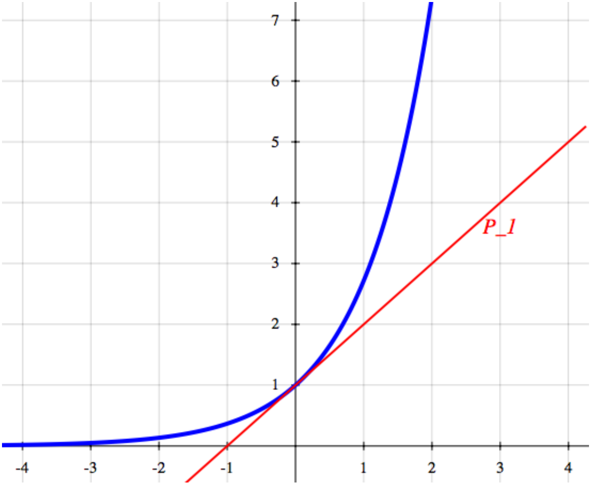
In one of the videos you saw that to prove f is analytic near a, it suffices to prove that

$$\lim_{n\to\infty} R_n(x) = 0$$

for all x near a.

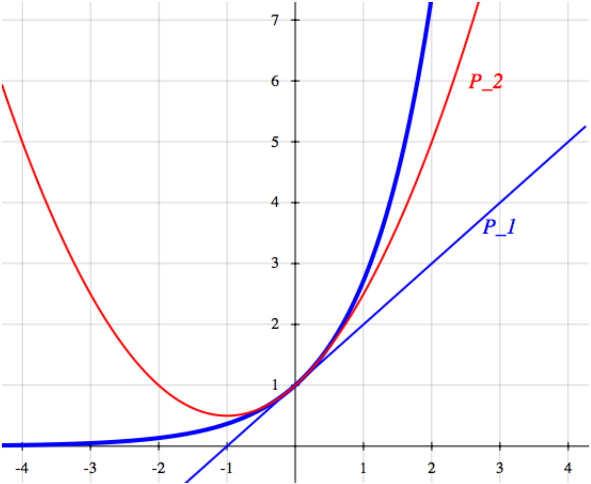
Here's what this looks like

The exponential function $f(x) = e^x$, and its first Taylor polynomial P_1



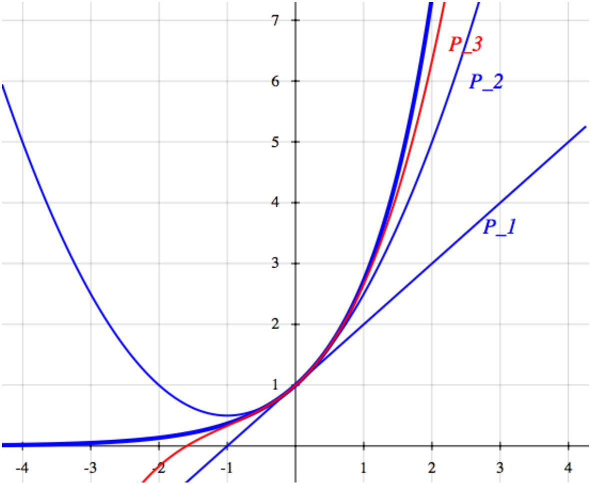
Here's what this looks like

The exponential function $f(x) = e^x$, and its second Taylor polynomial P_2



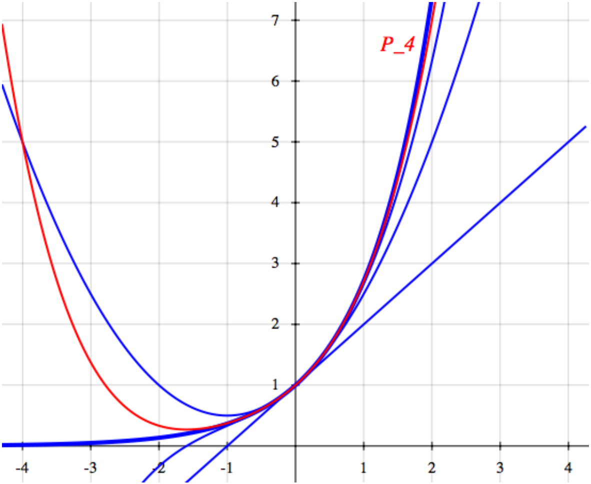
Here's what this looks like

The exponential function $f(x) = e^x$, and its third Taylor polynomial P_3



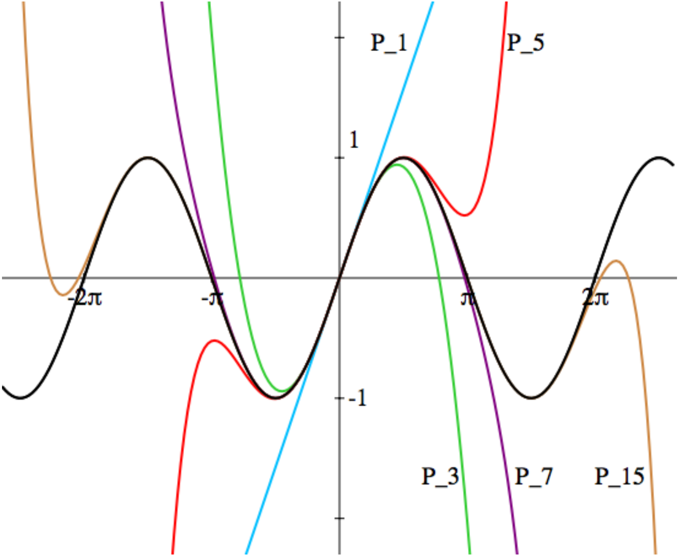
Here's what this looks like

The exponential function $f(x) = e^x$, and its fourth Taylor polynomial P_4



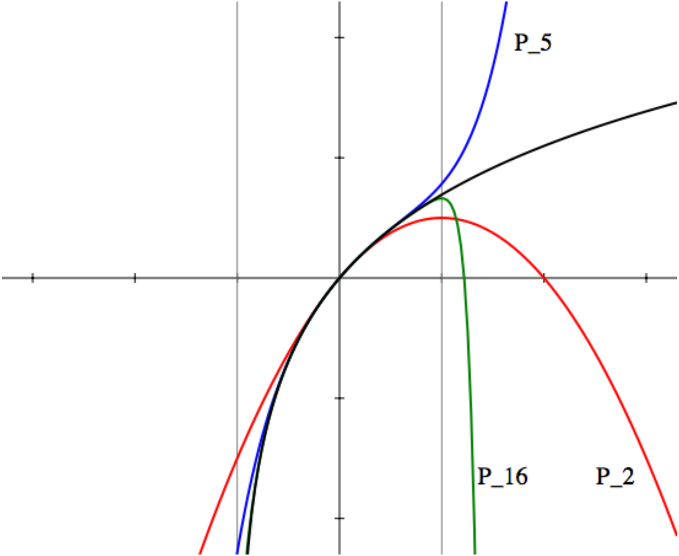
Some more pictures

Here is a graph of $\sin(x)$, and some of its Taylor polynomials.

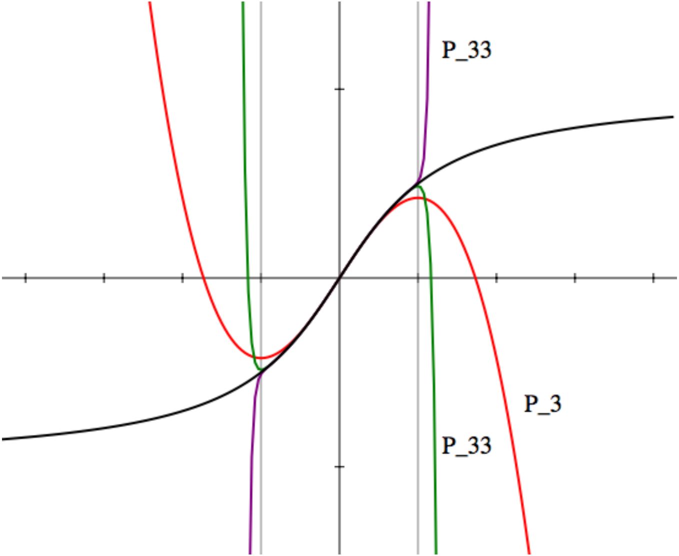


Some pictures

Here is a graph of $\log(1 + x)$, and some of its Taylor polynomials.

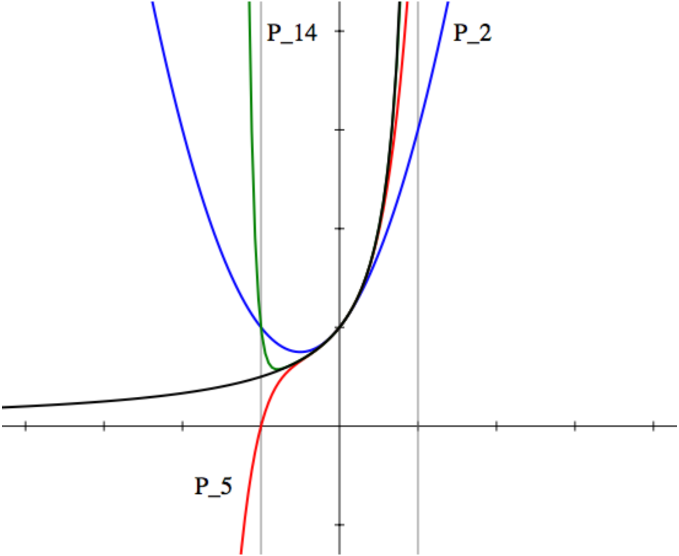


Here is a graph of arctan(x), and some of its Taylor polynomials.



Some pictures

Here is a graph of $\frac{1}{1-x}$, and some of its Taylor polynomials.



Some definitions

Now, to be more precise about what we mean by analytic:

Definition

Let f be a function defined at least on an open interval I.

- f is $\underline{C^1 \text{ on } I}$ of f' exists and is continuous on I.
- More generally, f is $\underline{C^n \text{ on } I}$ if $f', f'', f''', \ldots, f^{(n)}$ all exist and are all continuous on I.
- f is $\underline{C^\infty \text{ on } I}$ (or sometimes smooth on I) if f has all of its derivatives at every point of I.
- f is analytic on I if for every $a \in I$, the Taylor series of f centred at a converges to $f(x)$ for all x near a.

These properties are listed in increasing order of strength.

These properties really are all different

You know (or will soon know) that e^x, $\sin(x)$, and $\cos(x)$ are all analytic on $\mathbb{R}$.

We showed that

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is C^∞ on $\mathbb{R}$, but not analytic on any interval containing 0.

For each k, the function $f(x) = |x|^{k+1}$ is C^k but not C^{k+1}.

The function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable everywhere, but its derivative is not continuous, so it is not C^1.

Taylor's Theorem

In order to show a function is analytic, we need a way of "getting a handle on" the remainder term $R_n(x)$. There are (at least) three famous results for doing this.

The most famous of them is below, and it gives an explicit expression for the remainder:

Theorem

Suppose f is C^{n+1} on an interval I that contains a point a. Let P_n be its n^{th} Taylor polynomial at a.

Then for all $x \in I$, we have $f(x) = P_n(x) + R_n(x)$, where

$$R_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt.$$

This integral is a mess though, so instead of working with it we use theorems that estimate its value

Lagrange's Remainder Theorem

A much more useful one, which you saw used to prove that e^x is analytic in one of the videos, is the following theorem due to Lagrange:

Theorem

Let f be C^{n+1} on an interval I containing a point a.

Then for any $x \in I$, we have:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some number c in between a and x.

This is a consequence of the MVT.
Note that the value of c depends on <u>both</u> n <u>and</u> x.

The easiest way to use this theorem is to put an upper bound M on $\left|f^{(n+1)}(c)\right|$, so you don't have to care about the particular c.

$\sin(x)$ is analytic

In this exercise you'll prove that $\sin(x)$ is analytic on $\mathbb{R}$.

1. (A warm up, just to set a goal.) Fix a real number a, and write down the Taylor series for $\sin(x)$ centred at a.

2. Fix an $x \in \mathbb{R}$ and a non-negative integer n, and use Lagrange's theorem to write down an expression for the remainder $R_n(x)$.

3. Find a positive number M such that $|f^{(n+1)}(c)| < M$, no matter what c is.

4. Prove that $\lim_{n\to\infty} |R_n(x)| = 0$.

5. Prove that $\lim_{n\to\infty} R_n(x) = 0$.

We conclude that $\sin(x)$ is analytic on all of $\mathbb{R}$!

Other analytic functions

So far you know (or may now assume that) these four functions are analytic:

$$e^x \qquad \sin(x) \qquad \cos(x) \qquad \frac{1}{1-x}.$$

(The first three being analytic on $\mathbb{R}$, and the last being analytic on $(-1,1)$.) All polynomials are also analytic, of course.
You also know that sums, products, compositions, derivatives, and antiderivatives of analytic functions are analytic.

Exercise: Prove that e^{-x^2} is analytic on $\mathbb{R}$. Derive its Taylor series.

Exercise: Prove that $\log(1+x)$ is analytic on $(-1,1)$. Write down its Taylor series (this was in one of the videos, but you should now be able to derive it easily).

Exercise: Prove that $\arctan(x)$ is analytic on $(-1,1)$. Derive its Taylor series.

Some insights

Once you know about Taylor series, the "dominant behaviours" of functions near the points at which the series are centred becomes very clear.

Example: In your tutorial on improper integrals, you were asked to think about this integral:

$$\int_0^1 \frac{\arctan(x)}{x^{1.1}}\, dx.$$

Now we know that

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1} \quad = \quad x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots \qquad \text{for } x \in (-1, 1).$$

Knowing this, what is the "dominant behaviour" of this function near 0?

New Taylor series from old ones

Please complete these two exercises for the next lecture.

Exercise: Compute the Taylor series of the following functions, at the specified centre points.

1. $x^5 \log(1 + x^3)$, about 0.
2. $e^x(1 - x^2)$ about 0.
3. $\cos^2 x$, about 0. (Hint: use a trig identity.)
4. $\sin(\pi x)$, about $\frac{1}{2}$. (Don't overthink this one.)
5. $\frac{1}{x}$, about a fixed, nonzero real number a.

Exercise: For each part of the previous question:

Call the function in question f, and let a be the point at which the series is centred. Find the value of $f^{(100)}(a)$.