MAT137 - Term 2, Week 10

- Problem Set 9 is today, by 11:59pm. **Don't leave the submission process to the last minute.**
- Today we will:
  - Talk about alternating series and the ratio test.
  - Talk about power series.
  - Start talking about Taylor polynomials.
- For next week:
  - Watch the next seven videos on Playlist 14 (14.5 through 14.11).
- I've included a few exercises in here that we didn't get a chance to discuss in class. They're indicated in red wherever they appear.

Exercises I left you with from last class.

Exercise: Suppose $\sum_{n=0}^{\infty} a_n$ is a convergent series with non-negative terms.

1. Does $\sum_{n=0}^{\infty} (a_n)^2$ necessarily converge or diverge? Is the converse true?
2. Does $\sum_{n=0}^{\infty} \frac{a_n+3^n}{a_n+7^n}$ necessarily converge or diverge?

Exercise: Suppose $\{a_n\}$ is a sequence with positive terms, and such that $\lim_{n\to\infty} a_n = \infty$.

Prove that $\sum \frac{1}{a_n^n}$ converges.

Absolute Convergence

So far most of our results have been about series with positive (or at least non-negative) terms. This is not by accident.

Definition

A series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ converges.

This notion is strictly stronger than regular convergence, as the next result illustrates.

Theorem

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

ie. Absolutely convergent series are convergent.

Absolute convergence

Note that a consequence of this is that a convergent series with positive terms is absolutely convergent. This is a deceptively powerful fact, as you will see later in the course.

Example: $\sum \frac{(-1)^n}{n}$ is convergent (a fact we can't prove easily yet, but will prove soon).

On the other hand, $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$, which we know diverges.

(The first series above is called the alternating harmonic series.)

Definition

A series which is convergent but not absolutely convergent is called conditionally convergent.

Alternating series

The most interesting series that involve negative terms are the so-called alternating series. These are series in which every other term is negative.

Definition

Let $\{b_n\}_{n=0}^{\infty}$ be a sequence with positive terms.

Then the series $\sum(-1)^n\, b_n$ is called an alternating series.

These sorts of series will come up very often when we talk about Taylor series.

You have some experience working with sequences like this from Problem Set 8.

Alternating Series Test

The great thing about alternating series is that there's a very simple convergence test for them.

Theorem (Alternating series test (AST))

Let $\{b_n\}_{n=0}^{\infty}$ be an <u>eventually decreasing</u> sequence with <u>positive</u> terms.

Suppose also that $\lim_{n\to\infty} b_n = 0$.

Then $\sum (-1)^n \, b_n$ converges.

Warning: The converse isn't true! Of course, if $\sum (-1)^n \, b_n$ converges we must have that $\lim_{n\to\infty} b_n = 0$. But it isn't necessarily true that b_n is eventually decreasing.

Alternating Series Test exercises

This test is pretty easy to use, so there isn't much to learn from doing lots of exercises. We'll just see a few quick ones.

Exercise: Determine whether or not the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+7}}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$
3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3n^2+1}{7n^2+3n+1}$
4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{n}}{n+4}$

Exercise: Construct a series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ such that:

- The series is convergent.
- $\{b_n\}$ is not monotonic.

Estimations based on alternating series

I skipped these two slides in the interests of time, but these things are worth knowing.

Another very useful feature of alternating series is that they allow us to estimate some things easily. In video 13.14 you saw a proof of the following fact:

Theorem

Let $\sum_{n=0}^{\infty}(-1)^n b_n$ *be an alternating series such that:*

1. $\{b_n\}_{n=0}^{\infty}$ *is decreasing; and*
2. $\lim_{n\to\infty} b_n = 0.$

Then the series converges by the AST. Let L be its value.

Let $\{S_n\}_{n=0}^{\infty}$ *be the sequence of partial sums of the series.*

Then for all $n \in \mathbb{N}$, $|S_n - L| < b_{n+1}$.

Estimations based on alternating series

I skipped these two slides in the interests of time, but these things are worth knowing.

In words, this theorem says that the sum of the first n terms of a convergent alternating series with decreasing terms differs from from the sum of the whole series by at most the absolute value of the $(n+1)^{\text{st}}$ term.

Exercise: Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

This series clearly converges by the AST. Call its sum L. Estimate the value of L to within an error of 10^{-6}.

Some numbers that may be useful for you:

$$7! = 5\,040, \quad 8! = 40\,320, \quad 9! = 362\,880, \quad 10! = 3\,628\,800$$

Surprising fact: $L = \frac{1}{e}$.

The Ratio Test

Theorem (Ratio Test)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with positive terms.

Suppose also that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists, and equals a real number λ.

Then:

1. *If $\lambda < 1$, then the series $\sum a_n$ converges.*
2. *If $\lambda > 1$, then the series $\sum a_n$ diverges.*
3. *If $\lambda = 1$, we can't conclude anything about $\sum a_n$.*

Note: Note that $\lambda = 0$ is fine here, in contrast to the LCT. Don't mix them up!

Exercises

We now have all of the convergence tests we're going to need. So, determine whether the following series converge or diverge.

1. $\displaystyle\sum \frac{1000^n}{n!}$.
2. $\displaystyle\sum \frac{(-1000)^n}{n!}$.
3. $\displaystyle\sum \frac{(2)(4)(6)\cdots(2n)}{(n+2)!}$.
4. $\displaystyle\sum \frac{\log(n)}{n^2}$.
5. $\displaystyle\sum \frac{\sin(n)}{n^2}$
6. $\displaystyle\sum \frac{\pi\, n^7 \log(n^2)}{7^n}$.
7. $\displaystyle\sum \frac{(2n)!}{(n!)^2}$
8. $\displaystyle\sum \frac{n!}{e^{n^2}}$

A new test

I skipped this slide in class in the interests of time. I still suggest thinking about this though!

To prove the Ratio Test, the idea is to compare a series $\sum a_n$ with a geometric series.

For any geometric series $\sum b_n$, where $b_n = r^n$ for some r, we have that

$$\frac{b_{n+1}}{b_n} = r,$$

and that $\sum b_n$ converges if and only if $|r| < 1$. The rest of the ratio test is basically the same idea.

Another way to characterize this property of geometric series is as follows:

$$\text{For every } n \in \mathbb{N}, \quad \sqrt[n]{b_n} = \sqrt[n]{r^n} = r.$$

A new test

So we can use the same idea to create a new test.

Suppose $\{a_n\}$ is a positive sequence such that $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.

What can you say about $\sum a_n$ when L is greater than, less than, or equal to 1?

Definition of power series

Definition

Let $\{c_n\}$ be a sequence of real numbers, and let $a \in \mathbb{R}$.

A series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is called a power series.

More specifically, it's called a power series in $(x-a)$ or a power series centred at a.

The specific case of power series cented at $a=0$ will be what we study most carefully. These power series look like this:

$$\sum_{n=0}^{\infty} c_n x^n.$$

Power series as functions

The idea here is to define functions with power series. So you should think about defining a function f via something like:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

From what we know about geometric series, we see that we to be careful with the domain of this function. It's very possible for the series on the right to converge for some values of x and not for others.

We can say the following, not very useful fact:

The domain of a power series $\sum\limits_{n=0}^{\infty} c_n (x-a)^n$ is the set of all $c \in \mathbb{R}$ such that $\sum c_n (c-a)^n$ converges.

Power series as functions

As you saw in one of the videos though, we can say something much more concrete.

We'll state this result for power series centred at 0 first.

Theorem

- *If the power series* $\sum_{n=0}^{\infty} c_n x^n$ *converges at some* $c \neq 0$, *then it converges absolutely for all* x *such that* $|x| < |c|$.
- *If* $\sum_{n=0}^{\infty} c_n x^n$ *diverges at some* d, *then it diverges for all* x *such that* $|x| > |d|$.

Intervals of convergence

As a result of the previous theorem, the set of points on which a power series $\sum_{n=0}^{\infty} c_n x^n$ converges can only have three forms:

Case 1: The power series converges only when $x = 0$.

Obviously every series $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = 0$.

An example where it converges nowhere else might be $\sum_{n=0}^{\infty} n!\, x^n$.

For any $x \neq 0$, we have that

$$\lim_{n\to\infty} n!\, x^n \quad \text{does not exist by the Big Theorem.}$$

Intervals of convergence

Case 2: The power series converges **absolutely** at all real numbers.

We know an example of this already: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

We can prove this with the ratio test:

$$\left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right| = \frac{1}{n+1}|x| \to 0 \quad \text{for all } x \in \mathbb{R}.$$

The interesting case

Case 3: There is a positive real number R such that the power series $\sum a_n x^n$ converges absolutely when $|x| < R$, and diverges when $|x| > R$.

We will soon see that when $|x| = R$ in this case, many things can happen. For now let's ignore that part.

Definition

Associated to every power series $\sum a_n x^n$ is a radius of convergence.

1. In Case 1 from above, we say the radius of convergence is 0.
2. In Case 2 from above, we say the radius of convergence is ∞.
3. In Case 3 from above, we say the radius of convergence is R.

Your tool for calculating this R will almost always be the Ratio Test.

What about the endpoints?

Exercise: Compute the domains of convergence of the following four power series (ie. not just the radii; check the endpoints):

1. $\sum x^n$.
2. $\sum \frac{(-1)^n}{n} x^n$.
3. $\sum \frac{1}{n} x^n$.
4. $\sum \frac{1}{n^2} x^n$.

What about when $a \neq 0$?

If your power series is not centred at zero, so it looks like $\sum c_n (x-a)^n$ for some $a \neq 0$, all the same stuff is true if you replace x with $x - a$.

So a series like this can:

1. Converge only at $x = a$.
2. Converge absolutely at all real numbers.
3. Converge absolutely when $|x-a| < R$ for some R, and diverge when $|x-a| > R$. (And again, any combination of things can happen when $|x-a| = R$.)

This shows that the set on which a power series converges must be an interval centred at a. We call it the interval of convergence.

This is also why we call it a power series centred at a.

Exercise

Find the intervals of convergence of the following power series:

1. $\sum \frac{(-1)^n}{\sqrt{n}} x^n$.
2. $\sum \frac{3^n}{(3n)!} x^n$.
3. $\sum \frac{1}{n\,7^n} x^n$.
4. $\sum \frac{n^7}{e^n} (x-4)^n$.

Reminder: The usual method here is to use the ratio test to get the radius of convergence, then analyze the two endpoints separately.

Differentiation and integration

The idea we're moving towards with this topic is that power series are sort of like "infinite degree polynomials".

Our hope is that we can differentiate (and integrate) them exactly like they're polynomials.

For example, wouldn't it be nice if:

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} c_n\, x^n\right) = \sum_{n=1}^{\infty} \frac{d}{dx}\left(c_n\, x^n\right) = \sum_{n=1}^{\infty} n\, c_n\, x^{n-1}?$$

Then we could differentiate and integrate these functions very easily.

Differentiation and integration

The moral of the story is that most of the time we can do this:

<u>Inside</u> the interval of convergence (ie. when $|x| < R$), you can treat a power series like a polynomial.

To state this more precisely, there are two parts. If a power series converges absolutely at x, then

1. If you differentiate a power series term by term, then the resulting series converges.

2. The function defined by the resulting series is actually the derivative of the function defined by the series you started with.

Differentiation and integration

Here's the first part. Again, we'll only state this for power series centred at 0, but it applies to all power series with the appropriate shift.

Theorem

Let $\sum c_n x^n$ be a power series with radius of convergence R (which could be ∞).

1. *The power series $\sum \frac{d}{dx}(c_n x^n) = \sum n\, c_n\, x^{n-1}$ also converges absolutely when $|x| < R$.*

2. *The power series*

$$\text{“}\sum \left(\int c_n x^n \, dx \right)\text{”} = \sum \frac{c_n}{n+1} x^{n+1}$$

also converges absolutely when $|x| < R$.

Differentiation

Here's the second part, which is very surprising, and very useful.

We'll state it for derivatives first.

Theorem

Let $\sum c_n x^n$ be a power series with radius of convergence R, and define:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{for all } x \in (-R, R).$$

Then f is differentiable on $(-R, R)$, and

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}(c_n x^n) = \sum_{n=1}^{\infty} n\, c_n x^{n-1} \quad \text{for all } x \in (-R, R).$$

In fact, this immediately implies that f is differentiable infinitely many times on $(-R, R)$.

Theorem

Let $\sum c_n x^n$ be a power series with radius of convergence R, and define:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \textit{for all } x \in (-R, R).$$

Then f is integrable on $(-R, R)$, and the function

$$F(x) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1} \quad \textit{is an antiderivative of } f \textit{ on } (-R, R).$$

You can do even more stuff

You can even multiply power series, though it's pretty tedious.

Suppose we define two functions with power series (centred at the same point):

$$f(x) = \sum_{n=0}^{\infty} b_n x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} c_n x^n$$

Also suppose that x is some point <u>inside</u> both of their intervals of convergence. Then:

$$f(x)g(x) = \left[b_0 + b_1 x + b_2 x^2 + \dots\right]\left[c_0 + c_1 x + c_2 x^2 + \dots\right]$$

You can expand and collect terms:

$$= b_0 c_0 + (b_1 c_0 + b_0 c_1)x + (b_2 c_0 + b_1 c_1 + b_0 c_2)x^2 + \cdots$$

and this series will also converge absolutely.

Absolutely convergent power series are great

This is the moral of the story:

<u>Inside</u> the interval of convergence (ie. when $|x| < R$), where the power series converges absolutely, you can treat a power series like a polynomial.

This is the main reason we care about absolute convergence so much.

Exercise

Recall that we know

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{for all} \quad |r| < 1.$$

1. Find a power series $\sum_{n=0}^{\infty} b_n x^n$ such that

$$\sum_{n=0}^{\infty} b_n x^n = \frac{1}{(1-x)^2}, \quad \text{for all} \quad |x| < 1$$

1. Find a power series $\sum_{n=0}^{\infty} c_n x^n$ such that

$$\sum_{n=0}^{\infty} c_n x^n = \arctan(x), \quad \text{for all} \quad |x| < 1$$