MAT137 - Term 2, Week 9

- Problem Set 9 is due next Thursday.
- Today we will:
  - Talk about convergence tests for series.
- For next week:
  - Watch the last two videos on Playlist 13.
  - Start watching the first few videos on Playlist 14. I'll get back to you on the weekend as usual about exactly how many you need to watch.
- Reminder: You have a few homework exercises on the last slide.

Some quick true/false questions

True or false?

1. If $\lim_{n\to\infty} a_n$ exists, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n$ exists.
3. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n + b_n$ both converge, then $\sum_{n=1}^{\infty} b_n$ also converges.
4. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, then $\sum_{n=1}^{\infty} a_n + b_n$ also diverges
5. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} (-a_n)$ both converge, then $\sum_{n=1}^{\infty} |a_n|$ must also converge.

Some more quick true/false questions

Let $\sum_{n=1}^{\infty} a_n$ be a series, and let $\{S_n\}_{n=0}^{\infty}$ be its sequence of partial sums.

True or false?

1. If $\{S_n\}_{n=0}^{\infty}$ is bounded and eventually monotonic, then $\sum_{n=0}^{\infty} a_n$ converges.
2. If $\sum_{n=0}^{\infty} a_n$ converges, then $\{S_n\}_{n=0}^{\infty}$ is eventially monotonic.
3. If $\sum_{n=0}^{\infty} a_n$ diverges, then there is an $n \in \mathbb{N}$ such that $S_n > 1000$

A quick geometric series exercise

Use what you know about geometric series to prove that $0.9999\cdots = 1$.

Hint: Write $0.9999\ldots$ as a geometric series.

The Integral Test

Theorem

Suppose f is a continuous, positive, decreasing function defined on $[1, \infty)$. Then

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges.}$$

NOTE: This doesn't say the series *equals* the integral.

Exercise: Determine whether the following series conververge or diverge:

$$\sum_{n=0}^{\infty} \frac{n}{e^n} \qquad \sum_{n=2}^{\infty} \frac{1}{n(\log(n))^2} \qquad \sum_{n=0}^{\infty} \frac{n}{n^2+1}$$

Most important consequence: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

The Basic Comparison Test

This convergence test for series is exactly the same as the BCT for improper integrals.

Theorem (Basic Comparison Test (BCT))

Suppose that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences with non-negative terms.

Suppose also that $a_n \leq b_n$ for all n.

1. *If $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.*
2. *If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.*

The Limit Comparison Test

This convergence test for series is exactly the same as the LCT for improper integrals.

Theorem (Limit Comparison Test (LCT))

Suppose that $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are sequences with* <u>*positive*</u> *terms.*

Suppose also that $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ *exists and equals a* <u>*positive*</u> *constant.*

Then

$$\sum_{n=0}^{\infty} a_n \text{ converges} \iff \sum_{n=0}^{\infty} b_n \text{ converges.}$$

Just as with the LCT for improper integrals, your strategy with this test will usually be to isolate the "dominant" behaviour of the sequence, and compare with that.

Your best tool for finding the dominant behaviour is the Big Theorem.

Convergence test exercises

Determine whether these series converge or diverge. If possible, see if you can prove your answers using more than one test.

1. $\sum \frac{1}{n+7}$
2. $\sum \frac{1}{n+\sqrt{n}}$
3. $\sum \frac{1}{7^n - n^7}$
4. $\sum \frac{7-\cos n}{ne^n}$
5. $\sum \left(\frac{1}{7} - \frac{1}{n}\right)^n$
6. $\sum \frac{7n^2+n+10}{\sqrt[3]{n^7+n^3}}$
7. $\sum \tan\left(\frac{1}{n}\right)$
8. $\sum \frac{1}{n^{1+\frac{1}{n}}}$

(By the way, why is it okay that I didn't include bounds on the sums?)

More convergence test exercises

Do these problems for homework. I gave hints for each of them at the end of lecture.

Exercise: Suppose $\sum_{n=0}^{\infty} a_n$ is a convergent series with non-negative terms.

1. Does $\sum_{n=0}^{\infty} (a_n)^2$ necessarily converge or diverge? Is the converse true?
2. Does $\sum_{n=0}^{\infty} \frac{a_n+3^n}{a_n+7^n}$ necessarily converge or diverge?

Exercise: Suppose $\{a_n\}$ is a sequence with positive terms, and such that $\lim_{n\to\infty} a_n = \infty$.

Prove that $\sum \frac{1}{a_n^n}$ converges.