MAT137 - Term 2, Week 8

- Problem Set 8 is due **today**, by 11:59pm. Don't leave the submission process to the last minute.
- Today's lecture will assume you have watched the first nine videos on playlist 13.
- Today we will:
  - Talk about series.

Remember improper integrals?

Almost everything we will learn about series this week and next week will be analogous to things we learned about improper integrals before reading break.

So we'll start by quickly reminding ourselves about some of these things.

First, recall that at the beginning of this term, we defined <u>definite integrals</u>, which look like this:

$$\int_a^b f(x)\,dx.$$

These things computed the area underneath the graph of a function on an interval of the form $[a, b]$.

Remember improper integrals?

We then took the definition of

$$\int_a^b f(x)\,dx$$

and asked what happens when b gets bigger and bigger.

More formally, we took a limit as b goes to ∞.

This defined a new concept, which didn't make sense before we defined it:

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\left[\int_a^b f(x)\,dx\right].$$

(We defined several other sorts of improper integrals as well, but we don't need those to understand series.)

Remember improper integrals?

Recall that the following two limits are very different things:

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{b\to\infty} \left[\int_a^b f(x)\, dx \right].$$

They shouldn't seem very similar, but in the context of series people confuse them all the time.

For example, it *feels* like in order for the improper integral on the right to converge, we need $f(x)$ to "get smaller and smaller" as x increases.

This isn't quite true, but it's good intuition.

Remember improper integrals?

But note that the opposite thing is not true.

We studied improper integrals of the form

$$\int_1^\infty \frac{1}{x^p}\,dx,$$

and saw that some of them converged and some of them diverged, despite the fact that

$$\lim_{x\to\infty} \frac{1}{x^p} = 0 \quad \text{for all } p > 0.$$

Remember improper integrals?

Some other results about improper integrals to remember:

Proposition

Let f be a **positive** *function defined on $[a, \infty)$ and integrable everywhere necessary.*

Then $\displaystyle\int_a^\infty f(x)\,dx$ either converges, or diverges to infinity.

Recall that this is true because if f is always positive, then

$$\int_a^b f(x)\,dx$$

must increase as b increases.

Remember improper integrals?

This is the first of two very important comparison tests we had for improper integrals:

Theorem (Basic Comparison Test (BCT))

Let $a \in \mathbb{R}$*, and let* f, g *be* **positive** *functions that are integrable on* $[a, b]$ *for every* $b > a$*.*

Suppose also that $f(x) \leq g(x)$ *for all* $x \in [a, \infty)$*. Then*

1. *If* $\int_a^\infty g(x)\,dx$ *converges, then* $\int_a^\infty f(x)\,dx$ *converges as well.*
2. *If* $\int_a^\infty f(x)\,dx$ *diverges, then* $\int_a^\infty g(x)\,dx$ *diverges as well.*

Remember improper integrals?

This is the second (and much more important) comparison test we had:

Theorem (Limit Comparison Test (LCT))

Let $a \in \mathbb{R}$, and let f, g be positive functions that are integrable on $[a, b]$ for every $b > a$.

Suppose also that $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and equals a positive constant.

Then:

$$\int_a^\infty f(x)\,dx \text{ converges} \iff \int_a^\infty g(x)\,dx \text{ converges}.$$

Series

Now that we've reminded ourselves about improper integrals, we can move on to series.

Series are defined exactly analogously to improper integrals of the sort we were thinking about earlier. We know how to add up finitely many numbers, so given a sequence $\{a_n\}_{n=1}^{\infty}$, we give a name to the sequence of partial sums:

$$S_k := \sum_{n=1}^{k} a_k = a_1 + a_2 + \cdots + a_k$$

and *define* the infinite sum to be the limit of these:

$$\sum_{n=1}^{\infty} a_n := \lim_{k \to \infty} S_k \quad \text{(if it exists).}$$

We say the series converges if this limit exists, and diverges otherwise.

Series

Determine whether the following series converge or diverge directly from the definion of series. If they converge, compute their values.

1. $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{n}{n+1}\right)$.
2. $\displaystyle\sum_{n=5}^{\infty} \frac{-3}{n^2-5n+4}$.
3. $\displaystyle\sum_{n=1}^{\infty} n$.

Recall the following formula:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

I didn't show this slide in lecture, but we saw this idea later anyway.

An important piece of intuition is that a sequence $\{a_n\}_{n=1}^{\infty}$ and its corresponding sequence of partial sums:

$$S_k := \sum_{n=1}^{k} a_k = a_1 + a_2 + \cdots + a_k$$

"contain the same information".
Obviously we know how to take a sequence and construct the sequence of partial sums, because that's the definition above.

Exercise: Go the other way! Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that its sequence of partial sums is:

$$S_k = \frac{1}{k+1}.$$

Reconstruct the original sequence $\{a_n\}$ from this information.

Facts: It doesn't matter where you start.

Something we know about sequences is that if you care about the limit of a sequence, it doesn't matter where you start.

In other words, $\{a_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=17}^{\infty}$ have the same limit (if it exists), and the fact that the first 16 terms of the latter sequence are "missing" doesn't matter.

The same is true of series, in the following sense. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence, and $M > 1$ is an integer. Then:

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \Longleftrightarrow \quad \sum_{n=M}^{\infty} a_n \text{ converges.}$$

Exercise: Prove this.

Facts: Linearity

Like **every other kind of limit** we've defined, there are some "limit law" type results about series:

Proposition

Suppose $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converge to* L *and* M*, respectively. Then:*

1. $\sum_{n=0}^{\infty} (a_n + b_n)$ *converges to* $L + M$.
2. $\sum_{n=0}^{\infty} (c\, a_n)$ *converges to* cL *for all* $c \in \mathbb{R}$.

These results both follow easily from what we know about finite sums of numbers, and the limits laws for sequences. The first one was proved in one of the videos.

Exercise: Prove the second one.

The "Necessary Condition Test"

This is an extremely fundamental result to the study of series.

Your intuition should be that if a series is going to converge, its terms should "get smaller", so that the partial sums get closer and closer together. This theorem formalizes this intuition.

Theorem (Necessary Condition Test (NCT))

Suppose $\{a_n\}_{n=0}^{\infty}$ is a sequence. If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

This theorem is most useful in its contrapositive form:

$$\text{If} \quad \lim_{n\to\infty} a_n \neq 0, \quad \text{then} \sum_{n=0}^{\infty} a_n \quad \text{diverges.}$$

Examples

The NCT allows us to easily determine that many series diverge. For example, all of the following series diverge because the limit of their terms is not zero:

- $\sum_{n=0}^{\infty} 7.$
- $\sum_{n=0}^{\infty} n.$
- $\sum_{n=0}^{\infty} \frac{n}{n+1}.$
- $\sum_{n=0}^{\infty} (-1)^n.$
- $\sum_{n=1}^{\infty} \log(n).$
- $\sum_{n=0}^{\infty} \sin(n).$

In the last case, it takes a bit of work to show that $\lim_{n\to\infty} \sin(n) \neq 0$, but it shouldn't be surprising.

A warning

WARNING

The converse of the NCT is not true.

That is:

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ it does not follow that } \sum_{n=0}^{\infty} a_n \text{ converges.}$$

Repeat this to yourself five times every day, until you begin saying it in your sleep.

The Harmonic Series

This is probably the single most important series we will see, and it demonstrates why the warning is necessary.

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series. It diverges.

I've actually shown you this series before, in disguise.

One of the first examples of recursively defined sequences I gave you was the sequence of Harmonic numbers:

$$H_1 = 1, \quad H_{k+1} = H_k + \frac{1}{k+1}.$$

(You can find this in my Lecture 6 slides.)

The Harmonic Series

After thinking about this a bit, we realized that:

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} + \frac{1}{k} = \sum_{n=1}^{k} \frac{1}{n}.$$

In other words, the Harmonic numbers are precisely the partial sums of the harmonic series.

When we first saw the Harmonic numbers, we proved that the sequence $\{H_k\}_{k=1}^{\infty}$ is unbounded by realizing they were equal to upper sums of the integral of $\frac{1}{x}$, and therefore $\lim_{k\to\infty} H_k$ does not exist.

That means that $\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{k\to\infty} H_k$ diverges.

Another result that caries over from integrals

Proposition

If $\{a_n\}_{n=0}^{\infty}$ is a sequence of positive numbers, then

$$\sum_{n=0}^{\infty} a_n$$

either converges, or diverges to infinity.

Exercise: Write a proof of this.

Hint: What can you conclude about the sequence of partial sums?

Okay, how can we evaluate series?

Having defined series convergence and having seen some series that diverge, how can we actually evaluate some convergent series?

In general, this is very difficult. Given some sequence $\{a_n\}_{n=0}^{\infty}$ it's usually hard to say anything meaningful about

$$s_n = \sum_{k=0}^{n} a_k.$$

Computing this is like the "series version" of computing indefinite integrals.

At this stage, there are essentially only two types of series whose values we can compute. We already saw some telescoping series earlier.

Geometric series

Given a real number r (r stands for "ratio"), the series

$$\sum_{n=0}^{\infty} r^n$$

is called a geometric series. These are series we can evaluate explicitly.

Recall the following result from one of the videos.

Theorem

The geometric series $\sum_{n=0}^{\infty} r^n$ *converges if and only if* $|r| < 1$.

In this case, $\sum_{n=0}^{\infty} r^n = \dfrac{1}{1-r}$.

Caution with geometric series

Just a quick warning about the previous result.

Earlier we said that where you start a series doesn't matter from the point of view of convergence.

It <u>does</u> matter for computing the actual value of a series though.

For example, the previous result says that $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \dfrac{1}{1-\left(\frac{1}{2}\right)} = 2$.

However, if we start the series from $n = 1$, we have:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \left[\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n\right] - 1 = 2 - 1 = 1.$$

Exercise Determine whether the following series converge, and if so compute their values:

1. $$\sum_{n=4}^{\infty} \frac{1}{7^n}.$$
2. $$\sum_{n=1}^{\infty} 3^{n+1}\, 7^{-n+2}.$$
3. $$\sum_{n=1}^{\infty} 3^{1/n}.$$
4. $$\sum_{n=1}^{\infty} \frac{4n^2 - n^3}{7 + 12n^3}.$$
5. $$\sum_{n=1}^{\infty} \frac{(-2)^{3n}}{7^n}.$$

Series computation exercises

You may not realize it, but you're already familiar with geometric-like series in at least one context.

Exercise: Let $\{a_n\}_{n=0}^{\infty}$ be the sequence that lists the digits of π. In other words:

$$a_0 = 3, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 1, \quad a_4 = 5, \quad a_5 = 9, \ldots$$

Write down a series (in terms of the sequence $\{a_n\}$) that *should* converge to π.

What we know about geometric series doesn't *strictly speaking* show that this series converges. Can you think of a convincing argument that it should converge? Using something like the comparison tests you learned for improper integrals maybe?