MAT137 - Term 2, Week 7

- Problem Set 8 is due on 1 March. Lots of time!
- Today we will:
  - Talk a little bit more about "The Big Theorem".
  - Talk about improper integrals.
- For next lecture I will ask you to watch some videos from Playlist 13, which is currently in the works. Check my webpage for details during reading week.
- Additional comments after lecture:
  - You have a homework question on slide 15.
  - There are some additional questions we didn't get a chance to talk about on slide 21. Think about them.

The Big Theorem

To remind you, here's the statement of The Big Theorem.

Theorem

For any positive number a, and any real number $c > 1$,

$$\log(n) << n^a << c^n << n! << n^n.$$

Note: I've used a natural logarithm in the statement above, but any base larger than 1 will work since for any $b > 1$

$$\log_b(n) = \frac{\log(n)}{\log(b)}.$$

Using the Big Theorem

I left you with two problems to think about from last class:

Exercise: Compute $\lim_{n\to\infty} \dfrac{7n^{12}\ \log_{88}(n^2)\ n!}{5(n+1)^{\pi}\ (3n)^n}$.

Another interesting question is whether you can "split" the gaps between the types of sequences in the Big Theorem. Doing this for all the gaps is a question on Problem set 8.

Exercise: Can you find a sequence $\{a_n\}_{n=1}^{\infty}$ such that:

$$\lim_{n\to\infty} \frac{n^c}{a_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n}{b^n} = 0,$$

for any positive real number c, and any $b > 1$?
In other words, can you find a sequence that grows much faster than any polynomial, but grows much slower than any exponential?

Improper integrals

Every definite integral we've discussed so far in this course has computed the (signed) area underneath the graph of a function f, on an interval of the form $[a, b]$ on which f is defined and bounded.

In this topic, we extend our ideas about bounded functions on closed intervals to open intervals, or unbounded functions.

For example, our methodology from before can't help us compute something like

$$\int_1^\infty f(x)\,dx,$$

since an approximation by rectangles would involve infinitely many rectangles, and we can't add up infinitely many numbers.

Improper integrals

Our solution will be to approximate the area we want with things we do know.

For example, suppose we want to compute the area under the graph of $\frac{1}{x^2}$ on the interval $[1, \infty)$.

We will "approximate" it by the area under the graph on an interval $[1, b]$ for some $b > 1$, then see what happens as we increase b.

Improper integrals

Definition

Let f be a function defined on an interval $[a, \infty)$ for some $a \in \mathbb{R}$, and such that f is integrable on $[a, b]$ for any $b > a$.

Then we define an improper integral by:

$$\int_a^{\infty} f(x)\, dx = \lim_{b \to \infty} \left[\int_a^b f(x)\, dx \right].$$

If this limit exists, we say that the improper integral converges, and that it diverges otherwise.

We can make a similar definition and say:

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a \to -\infty} \left[\int_a^b f(x)\, dx \right].$$

Example

Example: Determine whether $\int_1^\infty \frac{1}{x^2}\,dx$ converges, and compute its value if it does.

By definition, we have

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{1}{x^2}\,dx\right] = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} -\frac{1}{b} + 1 = 1.$$

The most important family of improper integrals

This result generalizes the result of the previous exercise, and gives us a very important result we will use many times for the rest of the year.

Question: For which $p > 0$ does the integral $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

Theorem

The improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges for all $p > 1$*, and diverges for all* $0 < p \leq 1$*.*

Question: Why didn't I mention values of $p \leq 0$?

Another example

Exercise: Determine whether $\int_0^{\infty} \cos(x)\,dx$ converges.

Solution: It doesn't, since $\lim_{b\to\infty} \sin(x)$ does not exist.

This leads us to notice a fact, mentioned in one of the videos and whose proof is more or less obvious given what we've done so far.

Theorem

Let f be a positive function defined on $[a, \infty)$ and integrable everywhere necessary.

Then $\int_a^{\infty} f(x)\,dx$ either converges, or diverges to infinity.

ie. If f is positive, it can't do what cosine did above.

Exercise

Exercise: Suppose f is continuous on $[1, \infty)$, and that $\lim_{x \to \infty} f(x) = 7$.

What can you conclude about $\int_1^\infty f(x)\,dx$? Try to prove your answer.

Exercise: Suppose f is continuous on $[1, \infty)$, and suppose you know that $\int_1^\infty f(x)\,dx$ converges.

What can you conclude about $\lim_{x \to \infty} f(x)$?

Integrals over all of $\mathbb{R}$

The same idea can be used to define integrals of the form $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$, but we have to be somewhat careful.

It seems like we might want to say:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b\to\infty}\left[\int_{-b}^{b} f(x)\,dx\right],$$

but this can get you into trouble.

Exercise: Using the "definition" above, compute the following three integrals:

$$\int_{-\infty}^{\infty} x\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} x+1\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} x-1\,dx$$

Integrals over all of $\mathbb{R}$

Definition

Let f be integrable on every interval of the form $[a, b]$ (in particular, this is true if f is continuous on $\mathbb{R}$). Then we say the improper integral

$$\int_{-\infty}^{\infty} f(x)\,dx$$

converges if

$$\int_{-\infty}^{1} f(x)\,dx \quad \text{and} \quad \int_{1}^{\infty} f(x)\,dx$$

both converge.

If they do, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ equals their sum.

Question: Is the choice of the number 1 in the definition above important?

Unbounded functions

We just saw how to extend our definition of integrability to unbounded intervals like $[a, \infty)$. What about functions that become unbounded at a point?

Question: What (if anything) is wrong with the following computation:

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -\frac{1}{1} + \frac{1}{-1} = -2.$$

Answer: $\dfrac{1}{x^2}$ is not integrable on $[-1, 1]$, so the definite integral here just doesn't make sense to write down.

However, $\dfrac{1}{x^2}$ *is* integrable on $[-1, -\epsilon]$ and $[\epsilon, 1]$ for any $\epsilon > 0$. So again we try to define integrals like this by approximating them.

Integrals of unbounded functions

Definition

Suppose f is defined on an interval $[a, b)$, but becomes unbounded near b. ie.

$$\lim_{x \to b^-} f(x) = \pm\infty.$$

Suppose also that f is integrable on $[a, x]$ for all $a < x < b$. Then we define:

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \left[\int_a^t f(x)\,dx \right].$$

As before, we say that the integral converges if this limit exists, and diverges otherwise.

As you can imagine, we can make an analogous definition when f becomes unbounded at the left endpoint of an interval.

Examples

Exercise: Compute the following two improper integrals:

$$\int_0^1 \frac{1}{x^2}\,dx \quad \text{and} \quad \int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

Homework Exercise: For values of $p > 0$ does $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converge? Prove your answer.

Comparison tests

Determining whether an improper integral converges or diverges can be hard.

We'd like to make it easier. In particular, we want to be able to use things we know about simple functions to say things about more complicated functions.

We have two powerful tools for doing this. These tools will be *very* useful for us when we talk about series as well, so pay close attention!

The Basic Comparison Test

Theorem (Basic Comparison Test (BCT))

Let $a \in \mathbb{R}$*, and let* f, g *be functions that are integrable on* $[a, b]$ *for every* $b > a$.

Suppose also that $0 \leq f(x) \leq g(x)$ *for all* $x \in [a, \infty)$*. Then*

1. *If* $\displaystyle\int_a^\infty g(x)\,dx$ *converges, then* $\displaystyle\int_a^\infty f(x)\,dx$ *converges as well.*
2. *If* $\displaystyle\int_a^\infty f(x)\,dx$ *diverges, then* $\displaystyle\int_a^\infty g(x)\,dx$ *diverges as well.*

Exercises

Exercise: Determine whether $\int_1^\infty \frac{1}{x+e^x}\,dx$ converges or diverges.

Note that we don't know how to find an antiderivative for this function, so you can't just apply the definition of improper integrals.

However, two inequalities seem to come to mind here. For $x \geq 1$, we know that

$$\frac{1}{x+e^x} \leq \frac{1}{x} \quad \text{and} \quad \frac{1}{x+e^x} \leq \frac{1}{e^x}.$$

Try both of them.

Exercise: Determine whether $\int_7^\infty \frac{1}{x-e^{-x}}\,dx$ converges or diverges.

Exercises

Exercise: Determine whether $\int_1^\infty \frac{1+7\sin^4(10x)}{\sqrt{x}}\,dx$ converges or diverges.

Trickier exercise: Determine whether $\int_1^\infty e^{-x^2}\,dx$ converges or diverges.

The Limit Comparison Test

This is the second (and much more important) comparison test you learned:

Theorem (Limit Comparison Test (LCT))

Let $a \in \mathbb{R}$, and let f, g be positive functions that are integrable on $[a, b]$ for every $b > a$.

Suppose also that $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and equals a positive constant.

Then:

$$\int_a^\infty f(x)\,dx \text{ converges} \iff \int_a^\infty g(x)\,dx \text{ converges}.$$

The analogous result is true for integrals that are improper at a point.

Extending the Limit Comparison Test

The first question a mathematician would ask is whether we can extend this result at all.

Exercise: Suppose f, g are positive, cts. on $[1, \infty)$, and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

- First, suppose $\int_1^\infty f(x)\,dx$ converges. Can you conclude anything about $\int_1^\infty g(x)\,dx$?
- Next, suppose $\int_1^\infty g(x)\,dx$ converges. Can you conclude anything about $\int_1^\infty f(x)\,dx$?

Hint: Try to take what you know about $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ and use it to find a relationship between $f(x)$ and $g(x)$.

Homework: Suppose f and g are continuous on $[1, \infty)$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$. What implications can you prove between the convergence of their improper integrals?

Using the Limit Comparison Test

The idea of the LCT is always to try to isolate the “dominant” behaviour of the integrand, and compare it to that.

Exercise: Determine whether $\int_1^\infty \frac{1 - e^{-x}}{x} \, dx$ converges or diverges.

Exercise: Determine whether $\int_1^\infty \frac{x^3 + 7x^2 + 3x + 1}{\sqrt{x^8 + 6x^4 + 10}} \, dx$ converges or diverges.

Trickier exercise: Determine whether $\int_0^1 \frac{\sin(x)}{x^{3/2}} \, dx$ converges or diverges.