MAT137 - Term 2, Week 6

- Your third term test is **tomorrow**. Please make sure you go to the correct room. Space is tight!
- Today we will talk about:
  - Sequences.
- For next week's lecture, please watch all the videos on playlist 12. Several of them are just examples which you don't *technically* need to watch... but watch all of them.

Sequences

To remind you of some definitions from the videos:

A *sequence of real numbers* is an infinite list of real numbers written in a specific order, like this:

$$a_1, a_2, a_3, a_4, \ldots$$

We call a_1 the "first term" of the sequence, a_7 the "seventh term" of the sequence, and so on.

We will sometimes start listing sequences at higher indices, like this:

$$a_7, a_8, a_9, a_{10}, \ldots$$

All that really matters is the order of the list, so we can start our indices from any number that's most convenient for us.

Sequences

Formally, a sequence should be thought of as a function.

Definition

Let k be a non-negative integer. A sequence of real numbers is a function $a : A_k \to \mathbb{R}$, where

$$A_k = \{ n \in \mathbb{Z} \,:\, n \geq k \}.$$

Rather than writing $a(n)$ for the n^{th} term of the sequence, we will almost always write a_n, as we did on the previous slide.

For example, consider the function $a : \mathbb{N} \to \mathbb{R}$ given by $a(n) = \frac{1}{n}$. Some of its values are:

$$a(1) = a_1 = 1, \quad a(2) = a_2 = \frac{1}{2}, \quad \ldots, \quad a(7) = a_7 = \frac{1}{7}, \quad \ldots$$

and so on.

Sequence notation

There are a lot of different notations people commonly use to write down sequences in compact forms. All of these are common ways of denoting the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

- $\{a_n\}_{n=1}^{\infty}$,
- $\{a_n\}$ for short,
- $\{a_n\}_{n\in\mathbb{N}}$
- $(a_n)_{n=1}^{\infty}$,
- (a_n) for short.

I will stick to using the first two.

Some examples of sequences

Some sequences look a lot like the functions we've been studying thusfar:

- $\left\{\dfrac{1}{2^n}\right\}_{n=1}^{\infty}$.
- $\left\{n^2+7n+1\right\}_{n=1}^{\infty}$.
- $\left\{\dfrac{e^n}{\sin(n)}\right\}_{n=1}^{\infty}$.
- $\left\{n^n\right\}_{n=1}^{\infty}$.

These all look like the sorts of functions we're familiar with, but with n's in place of x's. We'll return to this idea a bit later.

Some sequences don't look like that, such as $\{n!\}_{n=1}^{\infty}$.

Recursively defined sequences

Something entirely new that can happen with sequences is that they can be *recursively defined*. This is a sequence defined by

- Specifying the value(s) of the first term(s).
- Describing a rule that defines any later term in terms of the previous terms.

Certainly the most famous recursively defined sequence is the *Fibonacci sequence*, which was mentioned in one of the videos:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

For this sequence we specify that $f_1 = f_2 = 1$, and then say that for any $n > 2$,

$$f_n = f_{n-1} + f_{n-2}.$$

More recursive sequences

Example: Consider the sequence defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \log(a_n + 1).$$

Example: Consider the sequence defined by

$$b_1 = 1 \quad \text{and} \quad b_{n+1} = \sqrt[3]{b_n + 6}.$$

Example: Consider the sequence defined by

$$c_1 = 1 \quad \text{and} \quad c_{n+1} = 1 + \frac{1}{c_n + 1}.$$

Exercise: Write out the first few terms of each of these sequences.

Increasing and decreasing sequences

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ is called

- ...increasing if $a_{n+1} > a_n$ for all n (ie. the terms always get bigger).
- ...decreasing if $a_{n+1} < a_n$ for all n (ie. the terms always get smaller).
- ...non-increasing if $a_{n+1} \leq a_n$ for all n (ie. the terms never get bigger).
- ...non-decreasing if $a_{n+1} \geq a_n$ for all n (ie. the terms never get smaller).

A sequence satisfying any of the above four properties is called monotonic or monotone.

A better method, continued.

Determine whether the following sequences are monotonic.

- $7, 7, 7, 7, 7, 7, 7, \ldots$
- $\left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$
- $\{\sin(n)\}_{n=1}^{\infty}$.
- $\{sin(n) + n\}_{n=1}^{\infty}$.
- $\left\{ \frac{n^2}{2^n} \right\}_{n=1}^{\infty}$.

You should have noticed that the last sequence is not monotonic, but feels like it should be somehow. We say that sequences like this are <u>eventually monotonic</u>, or that a "tail" of the sequence is monotonic.

Bounded sequences

Another definition that works exactly the same as for functions:

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ is called

- ...bounded above if $\exists M \in \mathbb{R}$ such that $a_n \leq M$ for all n.
- ...bounded below if $\exists m \in \mathbb{R}$ such that $a_n \geq m$ for all n.
- ...bounded if it's bounded above and below.
- ...unbounded if it's not bounded.

Check your understanding: Convince yourself that every non-decreasing sequence is bounded below.

Some exercises

Exercise: Show that the sequence given by $a_n = n^{1/n}$ is bounded, and decreasing for $n \geq 3$.

Hint: Consider the function $f(x) = x^{1/x}$.

Exercise: Consider the sequence defined recursively by

$$b_1 = 1 \quad \text{and} \quad b_{n+1} = \sqrt[3]{b_n + 6}.$$

Show that this sequence is bounded and monotonic.

Hint: Prove both of them by induction.

Monotone sequences

True or false: Suppose f is an increasing function, and we define a sequence by $a_n = f(n)$. Then $\{a_n\}_{n=1}^{\infty}$ is necessarily an increasing sequence.

True or false: Suppose f is a function and the sequence $a_n = f(n)$ is increasing. Then f is necessarily an increasing function.

A nice example, that we'll see again later

Consider the sequence $\{H_n\}_{n=1}^{\infty}$ of "harmonic numbers" defined recursively as follows:

$$H_1 = 1 \quad \text{and} \quad H_{n+1} = H_n + \frac{1}{n+1}.$$

- Write down the first few terms of this sequence.
- Convince yourself that this sequence is increasing.
- Convince yourself that for each n,

  $$H_n > \int_1^{n+1} \frac{1}{x}\, dx.$$

  (draw a picture).
- From the previous part, convince yourself that $H_n > \log(n+1)$.
- From the previous part, convince yourself that $\{H_n\}$ is an unbounded sequence.

Continuing the analogy between sequences and the functions we've studied before, we now define the limit of a sequence. The definition is *exactly* analogous to the definition of a limit of a function at infinity.

Recall we had the following definition before:

Definition

We say that $\lim_{x \to \infty} f(x) = L$ when

$$\forall \epsilon > 0 \; \exists M \in \mathbb{R} \text{ such that } x > M \implies |f(x) - L| < \epsilon.$$

Here's our new definition:

Definition (Sequence convergence)

Let $L \in \mathbb{R}$. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to L if:

$$\forall \epsilon > 0 \; \exists M \in \mathbb{R} \text{ such that } n > M \implies |a_n - L| < \epsilon.$$

In this case, we write $\lim_{n \to \infty} a_n = L$ or sometimes simply $a_n \to L$.

If a sequence converges to some L, we say it is convergent.

If no such limit exists, we say it is divergent.

Convergence

Exercise: Show that $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

Exercise: Show that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ does not converge.

As you should suspect, these limits are closely connected to limits of functions at infinity in the following way:

Theorem

Let f be a function, L a real number, and suppose $\lim_{x\to\infty} f(x) = L$.

Define a sequence by $a_n = f(n)$. Then $\lim_{n\to\infty} a_n = L$ as well.

Proof.

Left as an easy exercise. □

Convergence

Exercise: Is the converse of the previous theorem true?

That is, if f is a function and $a_n = f(n)$ is a sequence defined from f in the usual way, and

$$\lim_{n \to \infty} a_n = L,$$

does it follow that

$$\lim_{x \to \infty} f(x) = L?$$

Can $\lim_{x \to \infty} f(x)$ exist *but not equal* L?

Limit laws

We won't list them all here, but the familiar limit laws you're aware of for functions also work here.

For example, if $a_n \to L$ and $b_n \to M$, then $a_n + b_n \to L + M$.

This is the analogue of the limit law for sums. The respective analogues are true for constant multiples, products, and quotients.

The Squeeze Theorem also works in this context, and its proof is essentially the same.

Two main theorems

There are two very important theorems about sequences you learned in the videos.

These results describe how boundedness, monotonicity, and convergence relate to one another.

Theorem

Every convergent sequence is bounded.

The contrapositive of this theorem is useful enough to be worth stating:

Remark

Every unbounded sequence diverges.

Monotone sequence theorem

The second theorem is more important, and more "deep".

Theorem (Monotone Sequence Theorem)

If $\{a_n\}_{n=1}^{\infty}$ *is a bounded, monotonic sequence, then it converges.*

More specifically, if the sequence is non-decreasing, then it converges to $\sup \{ a_n : n \in \mathbb{N} \}$.

Similarly, if the sequence is non-increasing, then it converges to $\inf \{ a_n : n \in \mathbb{N} \}$.

Using the MST

Note that the MST doesn't really help you compute limits. It just tells you limits exist sometimes.

It *does* tell you what value the limit should equal (a supremum or infimum), but computing that value usually amounts to doing the same work as proving the limit would involve.

The MST is a very important theorem, and its value is largely theoretical. On the next slide we'll see an example of its use.

Using the MST

Recall that we proved earlier that the sequence defined recursively by

$$b_1 = 1 \quad \text{and} \quad b_{n+1} = \sqrt[3]{b_n + 6}.$$

is bounded and monotonic.

What can we conclude about this sequence now?

Suppose it converges to L. Take the limit of both sides of the equation that defines the sequence:

$$b_{n+1} = \sqrt[3]{b_n + 6}.$$

What do you get? Be sure to explicitly justify every step of your computation.

You really need the MST there

Without knowing the sequence $\{b_n\}$ converges (by the MST), the proof above doesn't work. Here's an example to illustrate that.

Recursively define a sequence by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = 1 - a_n.$$

Apply the same methodology from the previous example to this. In other words, assume the limit exists and equals L, and take a limit of both sides. What do you get?

This all seems fine, but if you pay attention to the actual sequence, its values are

$$1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

which obviously doesn't converge.

The Big Theorem

Of all the things we learn about sequences, this will probably be the most useful.

This theorem allows us to compute many limits very easily.

For example, recall this sort of limit from the first term:

$$\lim_{x\to\infty} \frac{7e^x + 12x^4 + \pi\log(x)}{10x^7 + 2e^x}.$$

This looks complex, but we learned that the exponential function $f(x) = e^x$ grows much faster than any polynomial or any logarithm. That is, we were able to prove (with l'Hopital's rule) that:

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0 \text{ for any positive } n, \text{ and} \qquad \lim_{x\to\infty} \frac{\log(x)}{e^x} = 0.$$

The Big Theorem

Using these two facts, computing the scary-looking limit from before is easy:

$$\lim_{x\to\infty} \frac{7e^x + 12x^4 + \pi\log(x)}{10x^7 + 2e^x} = \lim_{x\to\infty} \frac{e^x\left(7 + 12\frac{x^4}{e^x} + \pi\frac{\log(x)}{e^x}\right)}{e^x\left(10\frac{x^7}{e^x} + 2\right)} = \frac{7}{2}.$$

The sort of calculation we did above works just as well for sequences involving logarithms, exponentials, and polynomials. So for example, the same proof will show that:

$$\lim_{n\to\infty} \frac{7e^n + 12n^4 + \pi\log(n)}{10n^7 + 2e^n} = 0.$$

The "Big Theorem" generalizes this result a bit, to take into account sequences like $n!$ and n^n that either aren't possible with functions, or rarely come up with functions.

Some notation before The Big Theorem

Recall this notation that was introduced in one of the videos:

Definition

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of *positive* numbers.

We say that a_n is much smaller than b_n, or b_n grows much faster than a_n, if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

If b_n grows much faster than a_n, we denote it by writing

$$a_n << b_n.$$

Computer scientists may be familiar with "little-o notation":

$$a_n << b_n \quad \iff \quad a_n \in o(b_n).$$

Preliminaries

Here's a rather obvious fact that we should get out of the way, by proving it:

Proposition

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be sequences of positive numbers.

If $a_n << b_n$ and $b_n << c_n$, then $a_n << c_n$.

Proof.

Exercise. □

A mathematician would express this result by saying "the relation $<<$ is transitive".

The Big Theorem

Theorem

For any positive number a, and any real number $c > 1$,

$$\log(n) << n^a << c^n << n! << n^n.$$

Note: I've used a natural logarithm in the statement above, but any base larger than 1 will work since for any $b > 1$

$$\log_b(n) = \frac{\log(n)}{\log(b)}.$$

Using the Big Theorem

Having proved this result, many horrific-looking limits are now essentially trivial to compute.

Exercise: Compute $\lim_{n\to\infty} \frac{7n^{12} \log_{88}(n^2)\, n!}{5(n+1)^{\pi}\,(3n)^n}$.

Another interesting question is whether you can "split" the gaps between the types of sequences in the Big Theorem.

Exercise: Can you find a sequence $\{a_n\}_{n=1}^{\infty}$ such that:

$$\lim_{n\to\infty} \frac{n^c}{a_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n}{b^n} = 0,$$

for any positive real number c, and any $b > 1$?

In other words, can you find a sequence that grows much faster than any polynomial, but grows much slower than any exponential?