MAT137 - Term 2, Week 5

- Reminder: Your seventh problem set is due **today**, by 11:59pm. Don't leave the submission process to the last minute! Try to start no later than 11pm.
- Reminder: Your third test is next Friday. Please **read the page about the test on the course website carefully**, specifically regarding time conflicts with other courses and tests.
- This lecture will assume you have watched all the videos on integration techniques.
- Today we're talking about:
  - Trigonometric substitution
  - Integrating rational functions
  - Volumes
- Before next week's lecture, please watch **all** videos on playlist 11.

Trigonometric substitutions

To remind you, our main tool here is the Pythagorean Identity, in the following three forms:

- $1 - \sin^2(x) = \cos^2(x)$.
- $1 + \tan^2(x) = \sec^2(x)$.
- $\sec^2(x) - 1 = \tan^2(x)$.

These will allow us to simplify many integrals that include terms of the following three forms:

- $a^2 - c^2x^2$.
- $a^2 + c^2x^2$.
- $c^2x^2 - a^2$

(where a and c are constants) by making substitutions which cause the left sides of the respective identities to appear.

Trigonometric substitutions

For example, if a term like $a^2 + c^2x^2$ appears in an integral, we may want to make the substitution

$$x = \frac{a}{c}\tan\theta,$$

so that this term will become

$$a^2 + c^2\left(\frac{a}{c}\tan\theta\right)^2 = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

after substituting.

This is the only new idea here. Everything else is just elaborate "bookkeeping".

Trigonometric substitutions

Exercise: Compute $\int \frac{\sqrt{9x^2 - 4}}{x}\, dx$.

First, take a moment to think about what the substitution should be, by thinking about what trig identity "fits" into the integrand somewhere.

You should have found the substitution $x = \frac{2}{3}\sec\theta$. Now compute $d\theta$ and plug everything into the interal and simplify as much as you can.

You should end up here:

$$\int 2\sqrt{\tan^2\theta}\, \tan\theta\, d\theta.$$

Carefully think about what we have to do with the $\sqrt{\tan^2\theta}$ part. What values can the x in the original integrand take? How does this affect the intervals on which our substitution is valid?

Trigonometric substitutions

We didn't discuss this example in detail in class, but I'm including it here so you can see it.

Another example, to illustrate how any quadratic function in an integral can be attacked with trig substitutions.

Exercise: Compute $\int \frac{x}{\sqrt{x^2 + 2x + 2}}\, dx$.

Here we don't see anything that looks like one of the Pythagorean identities. But it's there. Hiding.

First, complete the square in the quadratic in the denominator.

You should get:

$$x^2 + 2x + 2 = (x + 1)^2 + 1.$$

Trigonometric substitutions

Now our integral looks like:

$$\int \frac{x}{\sqrt{(x+1)^2+1.}}\, dx$$

The quadratic in the denominator *does* look like the identity $\tan^2 x + 1 = \sec^2 x$ now!

So make an appropriate trigonometric substitution, and compute the corresponding $d\theta$.

You should choose:

$$x + 1 = \tan\theta \quad \text{and so} \quad dx = \sec^2\theta\, d\theta \quad \text{and} \quad x = \tan\theta - 1.$$

Put all of this into the integral and simplify.

Trigonometric substitutions

Substituting these things into our original integral, we get:

$$\int \frac{\tan\theta - 1}{\sqrt{\tan^2\theta + 1}} \sec^2\theta \, d\theta = \int \frac{\tan\theta - 1}{\sqrt{\sec^2\theta}} \sec^2\theta \, d\theta.$$

Remember to think carefully about the $\sqrt{\sec^2\theta}$ part.

From here we can proceed as normal (assuming $\sec\theta$ is positive, just for simplicity):

$$= \int \frac{\tan\theta - 1}{\sec\theta} \sec^2\theta \, d\theta = \int \tan\theta \sec\theta - \sec\theta \, d\theta.$$

And we've successfully reduced the problem to an integral we can do.

Integrating rational functions

The idea of this section is to take a rational function (ie. a quotient of polynomials) and write it as a sum of simpler rational functions that are easy to integrate.

We all know how to take two rational functions and combine them into one, like this:

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)+3(x+1)}{(x+1)(x-3)} = \frac{5x-3}{x^2-2x-3}.$$

What we're trying to figure out with this topic is how to go the other way, so that we can transform a difficult integral into a simpler one we already know how to do:

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \;\ldots\; = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx.$$

Partial fractions decompositions

Exercise: The partial fractions decomposition of $\dfrac{x^2+1}{x^2(x-7)}$ is of the form:

$$\frac{A}{x^2} + \frac{B}{x-7}.$$

True or false?

Exercise: The partial fractions decomposition of the same function as above is of the form:

$$\frac{Ax+B}{x^2} + \frac{C}{x-7}.$$

True or false?

Exercise: The partial fractions decomposition of the same function as above is of the form:

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-7}.$$

True or false?

Exercise: Compute the following integral:

$$\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx.$$

Irreducible quadratics

What do we do when the denominator includes a quadratic we can't factor?

For example, consider the rational function $\frac{3x^2 - 5x + 4}{(x-1)(x^2+1)}$.

Would a decomposition like this one work?

$$\frac{3x^2 - 5x + 4}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x^2+1}$$

We don't have enough constants to do the job here. How can we fix this?

Exercise: Compute the following integral:

$$\int \frac{3x^4+4x^3-3x^2+6x+6}{(x-1)(x^2+1)}\,dx = \int \frac{3x^4+4x^3-3x^2+6x+6}{x^3-x^2+x-1}\,dx.$$

Applications of integration

There are an endless supply of applications of the techniques of integration in math and the sciences. In this course we're going to focus on one of them: using integration to compute volumes.

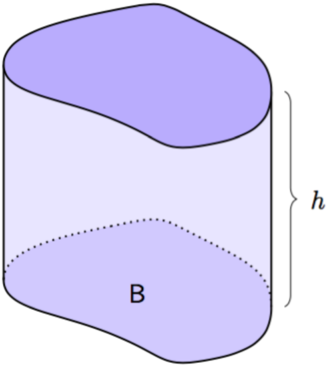
It's common for students to try to remember this material as a collection of formulas and nothing else, which is of course a bad idea.

The goal is to understand how the concept that underlies the definite integral (taking a "limit" of finer and finer approximations to something to get its exact value) can be applied to these situations.

Volumes

Recall how you compute the volume of a cylinder or triangular prism. In both cases you find the area of the "base" shape (a circle or triangle, respectively), and then multiply by the height.

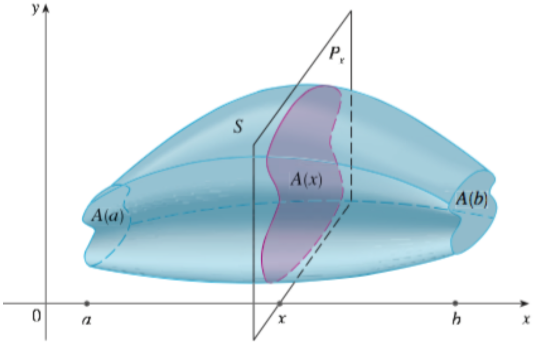
In the case of a more general solid like this, the idea is the same:



$$A = \text{area}(B) \cdot h$$

Volumes

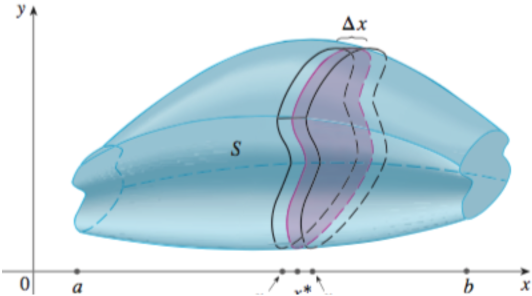
In the case of more complicated solids like this:



The situation is not as simple, but the methods of integration allow us to find the answer.

Volumes

Imagine partitioning up the interval $[a, b]$ into n pieces. In the i^{th} subinterval, pick a point x_i^*.



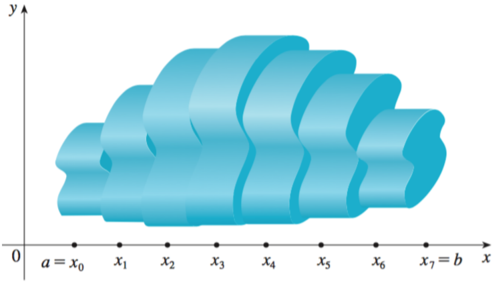
Approximate the volume of the shape over the i^{th} subinterval by assuming the cross-sectional area on it is always $A(x_i^*)$. Then the volume of this approximation over the i^{th} subinterval is $A(x_i^*)\Delta x_i$.

Volumes

If we do this for every subinterval, we get the following approximation of the volume V:

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i.$$

This corresponds to the volume of a shape like this:



Of course, the expression above should remind you of a Riemann sum.

The finer these partitions get, the closer this approximation should get to the actual volume V.

Accordingly, this means we should get:

$$V = \int_a^b A(x)\,dx$$

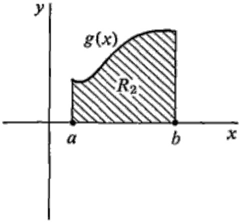
(where again $A(x)$ is the cross-sectional area of a slice of the solid taken at the point x).

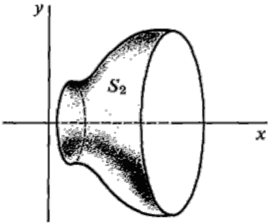
Example: Consider the solid whose base is the region bounded between $y = x$ and $y = x^2$, and whose cross-sections parallel to the y-axis are squares. Compute its volume.

Volume examples

The easiest examples of solids whose volumes you can compute in this way are "solids of revolution".

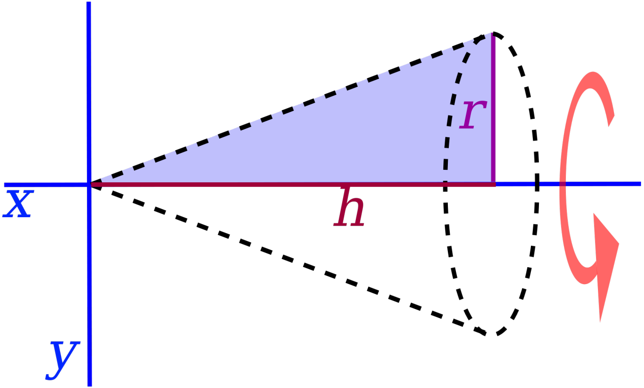
These are obtained by taking a region on the plane (often a region bounded between two curves), and rotating it about some line to obtain a solid.





Example: Find the volume of the right circular cone with base radius r and height h.

To use this method, we first realize the cone as a solid of revolution:



Volume examples

Example: Consider again the region bounded between $y = x$ and $y = x^2$. Rotate this region around the x-axis to form a solid. Compute its volume.

Example: Take the same region as above, but now rotate it around the y-axis. Compute the volume of the resulting solid.

Example: Use the same region again, but now rotate it about the line $x = -1$. Compute the volume of the resulting solid.

Another way of slicing things up

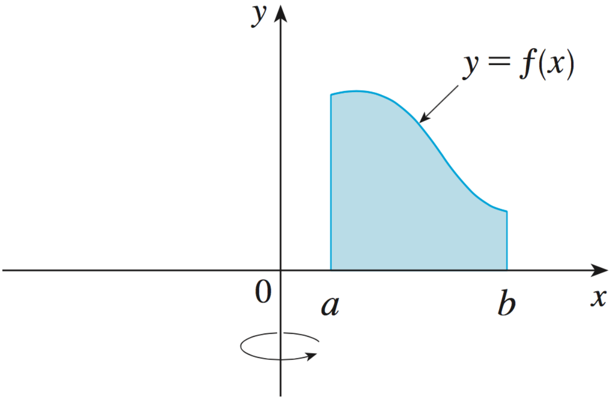
So far we've approximated the volumes of shapes by dividing them into a series of discs.

Another way to think of this is that we took our solid, and made a bunch of **straight** cuts in it, then approximated the volumes of the resulting pieces. Then we took a limit to get an integral.

Instead of straight cuts, we can make circular cuts. Think of an apple corer:

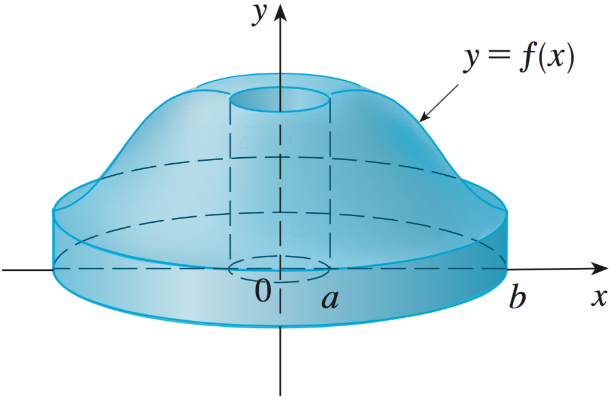
Another way of slicing

Suppose we start with a region like this:



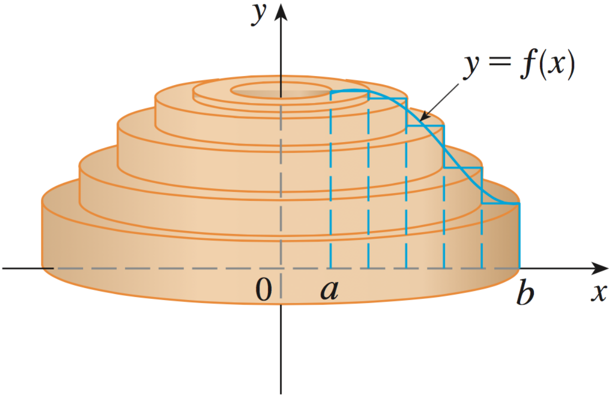
Another way of slicing

And rotate it about the y-axis to obtain a solid like this:



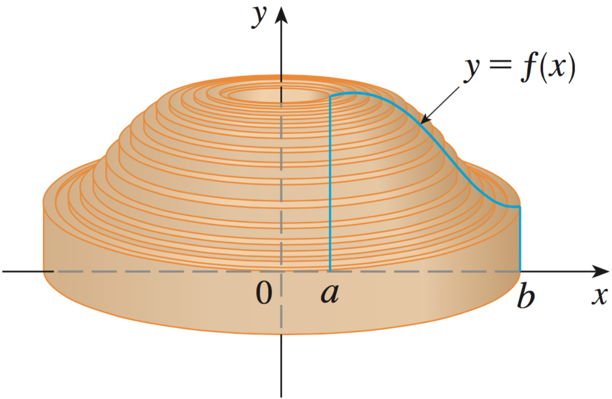
Another way of slicing

We can divide $[a, b]$ into five subintervals and get an approximation like this:



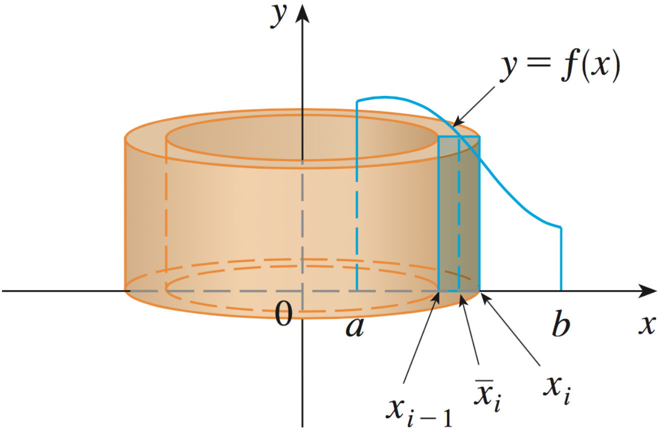
Another way of slicing

If we divide $[a, b]$ into many more subintervals, we get an approximation like this:



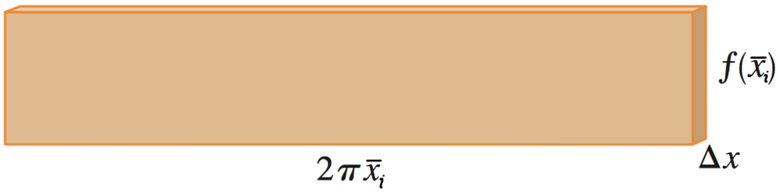
Another way of slicing

Going back to the coarser approximation, we can look at just one of the "slices":



Another way of slicing

Since we're imagining this slice being *very* thin, we can "unroll" it:



The volume of this shape is: $2\pi\overline{x_i} \cdot f(\overline{x_i}) \cdot \Delta x_i$.

Another way of slicing

Again, this should remind you of a Riemann sum.

If we do this for every slice, we obtain an approximation for the volume V:

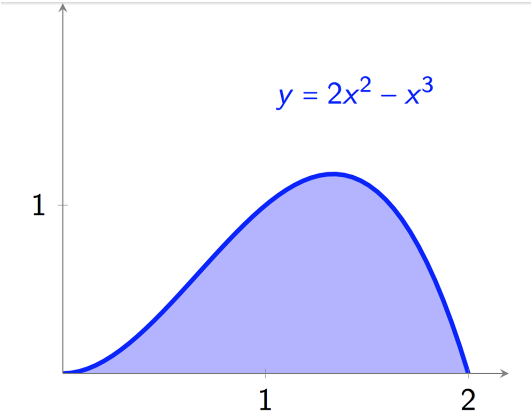
$$V \approx \sum_{i=1}^{n} 2\pi \overline{x_i} \cdot f(\overline{x_i}) \cdot \Delta x_i$$

This is a Riemann sum. This suggests the actual volume should equal:

$$V = \int_a^b 2\pi x\, f(x)\, dx.$$

Example

Consider the region between the curve $y = 2x^2 - x^3$ and the x-axis in the first quadrant. Rotate this region around the y-axis, and compute the volume of the resulting solid.



Example

Earlier you computed the volume of the solid obtained by rotating the region bounded between $y = x$ and $y = x^2$ about the y-axis.

Compute this volume again, using this new method of slicing up the shape.

Homework: Use either (or both!) of these methods to derive the formula for the volume of a sphere.