MAT137 - Term 2, Week 3

- Reminder: Your sixth problem set is due **today**, by 11:59pm. Please don't leave the submission process to the last minute!

- This lecture will assume you have watched **all** of the videos on the FTC, and at least the first video from playlist 9.

- Today we're talking about:
  - Properties of definite integrals
  - The FTC
  - Substitution

- Before next week's lecture, please watch up to video 9.14 on playlist 9. I'm not 100% sure if we'll cover all of it, but watch them all just in case. They're mostly short examples.

- Note that you have a problem to think about before next week's lecture. See the last slide.

Elementary properties of integrals

Reminder of some things from the videos:

- If $b < a$, then
$$\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx.$$

- Definite integrals are *linear*:
$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$
$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \text{ for any real number } c.$$

Elementary properties of integrals

This property is sometimes called "linearity of domain" or something similar:

For any numbers a, b, c,

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

In the case where $a < b < c$ at least, this is very intuitive from a picture.

You will prove this (the case in which $a < b < c$, at least) when you do Problem Set 6.

Example

Suppose f and g are integrable on $[0, 7]$, and you know the following things:

- $\int_0^3 f(x)\,dx = 4$
- $\int_3^5 f(x)\,dx = 1$
- $\int_5^7 f(x)\,dx = 2$
- $\int_2^5 f(x)\,dx = 3$
- $\int_0^7 f(x) + g(x)\,dx = 10$
- $\int_7^6 f(x)\,dx = -1$

Compute the following quantities:

- $\int_0^7 7f(x)\,dx$
- $\int_0^2 f(x)\,dx$
- $\int_5^6 f(x)\,dx$
- $\int_0^7 g(x)\,dx$

Monotonicity and Subnormality

Here are our two other useful properties. We won't use them directly today, but they're worth noting.

Theorem (Monotonicity)

If f and g are integrable on $[a, b]$, and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

The proof of this is left as an exercise. It should be pretty intuitive.

Theorem (Subnormality)

If f is integrable on $[a, b]$, then $|f|$ is also integrable on $[a, b]$, and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Antidervatives

A reminder of a definition from the videos. This definition is sort of like the "opposite" of a derivative.

Definition

Let f be a function defined on $[a, b]$.

Another function F is called an antiderivative of f on $[a, b]$ if

- F is continuous on $[a, b]$, and differentiable on (a, b)
- $F'(x) = f(x)$ for all $x \in (a, b)$.

So an antiderivative of f is essentially another function F whose derivative is f.

Properties of antiderivatives

We don't know much about antiderivatives yet, but from the definition alone there are many things we can say.

Theorem

Let F and G be antiderivatives of f and g on $[a, b]$, respectively, and let $c \in \mathbb{R}$. Then:

- *cF is an antiderivative of cf.*
- *$F + G$ is an antiderivative of $f + g$*

These follow immediately from the linearity of differentiation (ie. from the limit laws).

Properties of antiderivatives

Here's another very important one. We laid the groundwork for this fact while talking about the MVT last term.

Theorem

Suppose F and G are both antiderivatives of f on $[a, b]$.

Then F and G differ by a constant.

Proof.

Exercise! □

Note that this exercise says that antiderivatives are not unique. If a function has an antiderivative, it in fact has infinitely many antiderivatives.

Computing antiderivatives

In tutorial this week, you practised computing antiderivatives, by guessing and checking.

Guessing and checking is *by far* the most important "technique" for computing antiderivatives, and all the techniques you'll learn later are fancy ways of guessing and checking.

For now, some examples to remind yourself of how this goes. Find antiderivatives of the following functions, by guessing and checking.

1. $f_1(x) = 7x^3 + e^x$
2. $f_2(x) = \sin x - \sec^2(\sin x)\cos(x)$
3. $f_3(x) = \sqrt{7x}$
4. $f_4(x) = 7^x + \frac{12x}{1+x^4}$

Follow-up: Find the antiderivative F of $f_1(x) = 7x^3 + e^x$ that satisfies $F(0) = 0$.

Antiderivative notation

Let f be a continuous function defined on some interval $[a, b]$.

Question: The notation $\int f(x)\,dx$ represents:

1. A number.
2. A function.
3. A collection of functions.
4. None of the above.

Question: $\int f(x)\,dx$ and $\int f(\heartsuit)\,d\heartsuit$ mean exactly the same thing. True or false?

Defining functions with integrals

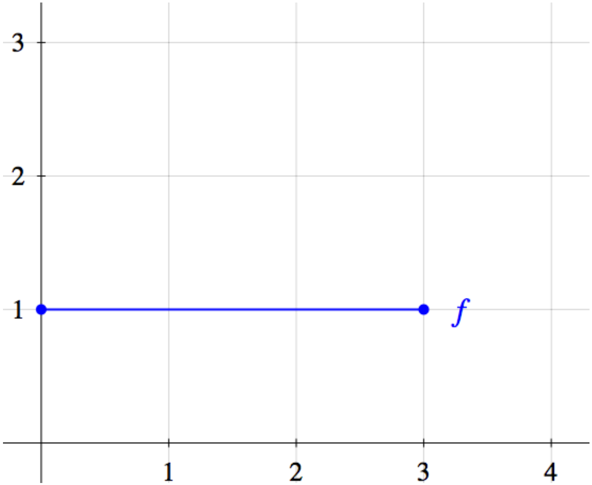
In order to understand the FTC, we need to understand a tricky way of defining a function.

Suppose f is an integrable function on $[a, b]$. Define a new function $g(x)$ in the following way:

$$g(x) = \int_a^x f(t)\, dt \quad \text{for } x \in [a, b].$$

First, convince yourself that the t inside the integral is just a "dummy variable". It only lives inside the integral. g is a function of x only.
This is much easier to understand as a picture, so be sure to draw one in your notes.

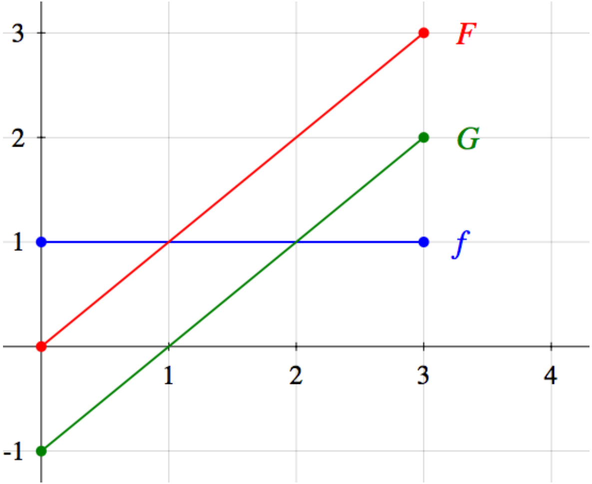
Exercise: Sketch the graphs of $F(x) = \int_0^x f(t)\,dt$ and $G(x) = \int_1^x f(t)\,dt$



For this example, you can actually derive formulas for these functions.

Defining functions with integrals

Exercise: Sketch the graphs of $F(x) = \int_0^x f(t)\,dt$ and $G(x) = \int_1^x f(t)\,dt$



Defining functions with integrals

Let f be an integrable function defined on an interval $[a, b]$. Let c and d be elements of $[a, b]$.

Question: Consider the following expressions.

1. $\displaystyle\int_c^x f(x)\,dx$
2. $\displaystyle\int_c^x f(u)\,du$
3. $\displaystyle\int_c^x f(c)\,dt$
4. $\displaystyle\int_c^x f(t)\,dt$

Which of these expressions make sense? Do any of them equal one another?

Question: Define two functions F_c and F_d on $[a, b]$ by:

$$F_c(x) = \int_c^x f(t)\,dt \quad \text{and} \quad F_d(x) = \int_d^x f(t)\,dt.$$

What is the relationship between the values of F_c and F_d?

The first FTC

My statement of this theorem is a little bit more general than the one given in the video.

Theorem (First Fundamental Theorem of Calculus)

Suppose f is integrable on $[a, b]$, and let $c \in [a, b]$. Then the function

$$F(x) = \int_c^x f(t)\,dt$$

is continuous on $[a, b]$.

F is differentiable at any point x where f is continuous, and at such a point $F'(x) = f(x)$.

The video only states the theorem for a continuous function f. In that case, F is simply an antiderivative for f on $[a, b]$.

Using FTC1

The main thing we use FTC1 for in this course is proving FTC2. But it does have some applications of its own.

Exercise: Find the derivative of $F(x) = \int_0^x \cos^2(t)\,dt$.

Exercise: Find the derivative of $G(x) = \int_x^7 7te^{t^2}\,dt$.

Trickier exercise: Find the derivative of $H(x) = \int_0^{x^2} \frac{\sin(t)}{t}\,dt$.

Hint 1: The answer is not $\frac{\sin(x^2)}{x^2}$.

Hint 2: Try to express $H(x)$ in terms of $F(x) = \int_0^x \frac{\sin(t)}{t}\,dt$.

Going further...

Exercise: Find the derivative of $F(x) = \int_{x^2}^{\sin(x)} \arctan(t)\, dt$.

Hint: Try to write $F(x)$ as a sum of two functions whose derivatives you know how to compute.

Okay! We're [trying to be] mathematicians, so it's time to generalize! Why solve one example, when we can solve **all** the examples?

Exercise

Let f, u, v be differentiable functions on $\mathbb{R}$, and define:

$$F(x) = \int_{u(x)}^{v(x)} f(t)\, dt.$$

Find a formula for $F'(x)$ in terms of some or all of f, u, v, f', u', v'.

The second FTC

The second FTC gives us an easy way to evaluate definite integrals. Easy compared to Riemann sums and stuff, at least.

Theorem (Second Fundamental Theorem of Calculus)

Let f be continuous on $[a, b]$. If F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Some notation

A reminder of some notation introduced in the videos.

To make our lives a bit easier, we will use the following notation:

$$F(x)\Big|_a^b = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

The first one was introduced in the video, and the second one is also common and means the same thing.

So for example:

$$7\sin(x^2)\Big|_0^3 = \Big[7\sin(x^2)\Big]_0^3 = 7\sin(3^2) - 7\sin(0).$$

Using FTC2 to compute definite integrals

Exercise: Compute $\int_0^{\pi} \sin(x)\, dx$.

Exercise: Compute $\int_1^2 3x^2 + 2x + 1$.

Thanks to FTC2, computing definite integrals reduces entirely to finding antiderivatives. We'll talk a lot about finding antiderivatives later, and develop some theory to help us do that.

But first, we'll talk a bit about areas.

Areas under curves

Question: Let f be a continuous function defined on an interval $[a, b]$. Then the quantity:

$$\int_a^b f(x)\ dx$$

represents the area between the graph of f and the x-axis. **True or false?**

Exercise: Using FTC2, compute $\displaystyle\int_0^{2\pi} \sin(x)\, dx$.

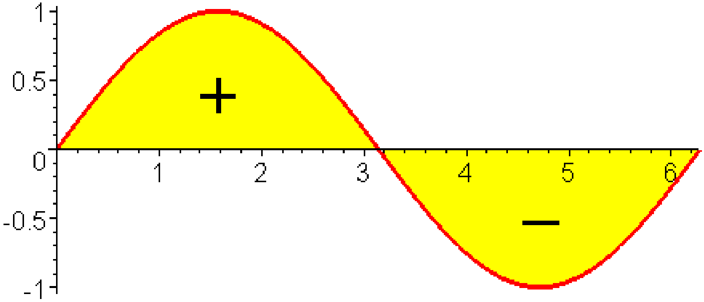
The answer you obtained was 0. But clearly there is area under the graph. So what's going on?

Signed area

The answer is that definite integrals actually compute something called signed area.

That means they assign a negative value to areas *underneath* the x-axis.

So this is what happend with the calculation you did above:



Total area

To compute the *total* (ie. *unsigned*) area between the graph $y = f(x)$ of a function and the x-axis, we "force" it to be above the x-axis, in the following sense:

Total area between the graph of $y - f(x)$ and the x-axis between a and b is:

$$\int_a^b |f(x)|\, dx.$$

You will explore this much more in next week's tutorial.

Substitution

Recall from the videos that the technique of substitution is derived from integrating the chain rule.

Here's the chain rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x).$$

Integrating this yields:

$$f(g(x)) + C = \int f'(g(x))\, g'(x)\, dx.$$

Substitution

$$f(g(x)) + C = \int f'(g(x))\, g'(x)\, dx.$$

To use this formula to compute the antiderivative $\int h(x)\, dx$, you must find two functions f' and g such that

$$h(x) = f'(g(x))\, g'(x).$$

Once you have found these functions, all you need to figure out is f (which is an antiderivative of f').

Sometimes this is easy, like in the case of

$$\int 2x(x^2 + 1)^7\, dx.$$

In this case $g(x)$ should be $x^2 + 1$ and $f'(x)$ should be x^7, and so $f(x) = \frac{x^8}{8}$.

Substitution exercises.

I left you with these problems (specifically the last one) to think about for next lecture. I intentionally jumped ahead to this.

Exercise: Consider the integral $\int e^{7x+5}\,dx$. What should our u be?

Exercise: Consider the integral $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$. What should our u be?

Trickier exercise: Compute $\int x\sqrt[3]{x+3}\,dx$.

You **must** do lots of examples to develop intuition for this. There's no *substitute* for experience. Make sure you do all the practise problems for this section (5.7).