MAT137 - Term 2, Week 2

- This lecture will assume you have watched **all** of the videos on the definition of the integral.

- Today we're talking about:
  - The definition of the integral.
  - Riemann sums.

- Before next week's lecture, please watch **all of the videos on playlist 8**. I will also ask you to watch **some videos from playlist 9**, which will be posted shortly. **Check my website over the weekend for details.**

Exercise from last class

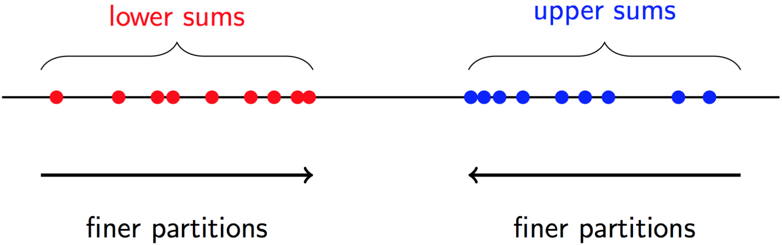
Last class, we computed the upper and lower sums for a constant function, and found that no matter which partition we took they were all equal to one another.

I also left you with this exercise:

Exercise: Show that if f is a *non-constant* function defined on $[a, b]$, then there exists a partition P such that $L_P(f) \neq U_P(f)$.

(Hint: This is *much* easier than it sounds. If you're doing anything tricky, you're overthinking it.)

Upper and lower sums



Upper and lower integrals

Let's remind ourselves of the definitions.

Suppose f is a bounded function defined on $[a, b]$.

Then the <u>upper integral</u> of f is:

$$\begin{aligned}\overline{I_a^b}(f) &:= \inf\{\text{upper sums of } f\}\\ &= \inf\{\, U_P(f) \;:\; P \text{ is a partition of } [a, b]\,\}.\end{aligned}$$

and similarly the <u>lower integral</u> is

$$\begin{aligned}\underline{I_a^b}(f) &:= \sup\{\text{lower sums of } f\}\\ &= \sup\{\, L_P(f) \;:\; P \text{ is a partition of } [a, b]\,\}.\end{aligned}$$

Upper and lower integrals

Remember that the upper sums are all *overestimates* of the area we're looking for. The upper integral acts like the "smallest" or "best" overestimate.

Similarly, the lower sums are all *underestimates* of the area we're looking for. The lower integral acts like the "largest" or "best" underestimate.

This is just intuition, but it's good intuition most of the time.

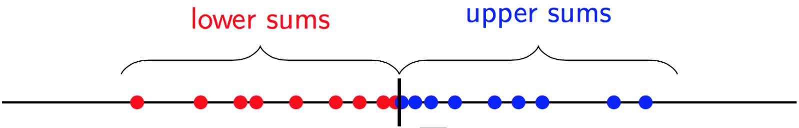
A function is called <u>integrable on $[a, b]$</u> if the "best" underestimate and "best" overestimate agree with one another.

Upper and lower integrals

You should try to keep these picture in mind.
A general (not necessarily integrable) function will result in this situation:

$$\underline{I_a^b}(f) \quad \leq \quad \overline{I_a^b}(f)$$

While an integrable function will give you this:



$$\underline{I_a^b}(f) = \overline{I_a^b}(f)$$

A tricky question

We didn't see this question in class, but it's a nice one to think about.

Let f be a bounded function on $[a, b]$. Are the following statements true or false?

1. There exists a partition P of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f) \quad \text{and} \quad \overline{I_a^b}(f) = U_P(f).$$
2. There exist partitions P and Q of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f) \quad \text{and} \quad \overline{I_a^b}(f) = U_Q(f).$$

These questions illustrate why the intuition from earlier isn't *quite* right.

The upper integral (for example) is the *supremum* of the set of upper sums, not necessarily the *maximum* of the set of upper sums.

Lower integral exercise

Recall the equivalent definition of supremum we found in an exercise last class:

Definition

If M is an upper bound of a set A, then M is the supremum of A if it satisfies the following:

$$\forall \epsilon > 0,\ \exists x \in A \text{ such that } M - \epsilon < x \leq M.$$

Exercise: The lower integral is the supremum of all the lower sums. Try to write a definition of the lower integral that's similar to the alternative definition above.

Hint: "$\forall \epsilon > 0$, there is a partition..."

Computing an integral, the absurdly hard way

In this exercise, you're going to do a lot of work to prove something very obvious. It's important that you do this once or twice in order to understand the definitions. This will be quite similar to the work done in video 7.8.

Consider the function $f(x) = \begin{cases} 0 & x = 0 \\ 7 & 0 < x \leq 1 \end{cases}$, defined on $[0, 1]$. Draw a picture of it in your notes.

1. Fix an *arbitrary* partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[a, b]$. What is $U_P(f)$?
2. What is the upper integral, $\overline{I_0^1}(f)$?
3. For the same fixed partition, what is $L_P(f)$? (Be sure to draw a picture of the rectangles!)
4. What is the lower integral, $\underline{I_0^1}(f)$?
5. What can you conclude about f?

The moral of the previous example

In the previous exercise, we were dealing with a function that was perfectly well behaved (ie. constant) everywhere except at one point.

The moral of the previous exercise is that we were able to "contain" the misbehaviour of the function inside as small an interval as we wanted. Outside that small interval, the function behaved very well and we could take care of it easily.

This is a very common and useful technique in proofs in calculus and analysis. If you continue through higher courses, you will see it very often.

In the next example, we'll use the same idea to talk about a much trickier function.

A much trickier example

By now you should believe that a function that is constant except for finitely many removable discontinuities is integrable. In this example we will deal with a function with *infinitely many* discontinuities.

Consider the function f defined on $[0,1]$ as follows:

$$f(x) = \begin{cases} 1 & \exists n \in \mathbb{N} \text{ such that } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

Before we do anything else, make sure you have a big, clear picture of what this function looks like.

(It's impossible to draw a perfectly accurate picture, but make sure you understand what it should look like.)

A much trickier example

$$f(x) = \begin{cases} 1 & \exists n \in \mathbb{N} \text{ such that } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

Let's deal with the lower sums and lower integral first.

1. Fix an *arbitrary* partition P of $[0, 1]$. Compute the lower sum $L_P(f)$.
2. Compute the lower integral $\underline{I_0^1}(f)$.

A much trickier example

$$f(x) = \begin{cases} 1 & \exists n \in \mathbb{N} \text{ such that } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

The upper integral is the tricky part, as you might expect. We will start slowly. Keep in mind the moral of the previous exercise.

1. Find a partition P of $[0, 1]$ containing exactly four points (including the endpoints), and such that $U_P(f) = \frac{1}{2} + 0.01$.
2. Find another partition Q of $[0, 1]$, this time containing exactly eight points, and such that $U_Q(f) = \frac{1}{4} + 0.0001$.
3. Prove that for any $\varepsilon > 0$, there exists a partition P of $[0, 1]$ such that $U_P(f) \leq \varepsilon$.
4. What is $\overline{I_0^1}(f)$?
5. What can we now conclude about f?

Riemann sums

Reminder of the definition (from the videos):

Let f be a bounded function on $[a, b]$. Let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a, b]$.

For each sub-interval $[x_{i-1}, x_i]$ created by the partition, choose any number in that sub-interval and call it x_i^*.

The number

$$S_P^*(f) = \sum_{i=1}^{N} f(x_i^*) \Delta x_i$$

is called a <u>Riemann sum for f and P</u>.

Note: There are many possible Riemann sums for the same f and P, since we can choose different x_i^*'s in each interval.

Riemann sums

Also recall that given a partition $P = \{x_0, x_1, \dots, x_N\}$ of $[a, b]$, we denote by $\|P\|$ the norm of the partition.

$\|P\|$ is the length of the *longest* sub-interval created by P.

Here's the point: If we *already know* f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} S_P^*(f).$$

This limit is confusing to calculate in general, so in practice we use simple sequences of partitions.

Partition exercises

In class, it seemed like everyone understood norms of partitions pretty well, so we skipped this exercise. If you're unsure about norms, spend a few minutes thinking about it.

Consider the interval $[0, 7]$. What are the norms of the following partitions?

1. $P = \{0, 1, 2, 3, 4, 5, 6, 7\}$.
2. $P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3, 4, 5, 6, 7\}$.
3. $P = \{0, 1, 2, 3, 7\}$
4. $P = \{0, 2, 4, 6, 7\}$
5. $P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, \frac{11}{2}, 6, \frac{13}{2}, 7\}$

Describe a simple sequence of partitions $P_1, P_2, P_3, \ldots$ such that $\|P_n\| \to 0$ as $n \to \infty$.

Using Riemann sums

Theorem

Let f be an integrable function defined on an interval $[a, b]$.

Let $P_1, P_2, P_3, \dots$ be a sequence of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} \|P_n\| = 0.$$

Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S^*_{P_n}(f).$$

Note that we don't need to specify which x_i^*'s we are using in each sub-interval of each partition. The result is true no matter which points we choose.

Example

Let's compute an integral using Riemann sums.

Consider the function $f(x) = x^2$ defined on $[0, 1]$. Is this function integrable?

(Recall: It is a theorem that if f is continuous on $[a, b]$, then it is integrable on $[a, b]$.)

For each $n \geq 1$, let P_n be the partition that splits $[0, 1]$ up into n equal sub-intervals. Write down this partition.

Fix an n and consider the partition P_n. For each i, what is Δx_i? What is $\|P_n\|$? What is $\lim_{n\to\infty} \|P_n\|$?

Example, continued

The fact that $\lim_{n\to\infty} \|P_n\| = 0$ means we can use this sequence of partitions to compute the integral.

Next we have to compute Riemann sums for these partitions. To do that, we need a point x_i^* in each sub-interval of each partition.

Let's use the right endpoints.

For a given P_n, give a formula for x_i^*, and then give a formula for $f(x_i^*)$.

Write down the Riemann sum $S_{P_n}^*(f)$ and factor out any unecessary terms from the sum.

Example, continued

We now need to compute:

$$\lim_{n\to\infty} S^*_{P_n}(f) = \lim_{n\to\infty} \left[\frac{1}{n^3} \sum_{i=1}^{n} i^2 \right].$$

Do to this, recall the following formula:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Use this formula to evaluate the the limit and obtain the value of the integral.

Homework exercise: Repeat the same steps, but instead choose the *left* endpoints for the x_i^* rather than the right endpoints. Verify that you get the same answer.