MAT137 - Week 12

- Last lecture of the term! Next Thursday is the last day of classes, but it "counts as a Monday".
- Your second test is **tomorrow, 4-6pm**. See the course website for details.
- Today's lecture is aboutl'Hôpital's rule, concavity, and curve sketching.

L'Hôpital's rule

Here's the statement of L'Hôpital's rule from last class.

Theorem

Let $a \in \mathbb{R}$, and let f and g be functions defined at and near a.

Suppose that

- $\lim_{x \to a} \frac{f(x)}{g(x)}$ *is indeterminate of type* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$.
- f *and* g *are differentiable near* a *(except possibly at* a*).*
- g *is never 0 near* a *(except possibly at* a*).*
- g' *is never 0 near* a *(except possibly at* a*).*
- $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ *exists, or is* $\pm\infty$.

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Other indeterminate forms

We saw that this theorem can help us deal with indeterminate limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

However, there are other indeterminate forms, all of which eventually reduce to one of the ones we've seen.

1. **Products of the form $0 \cdot \infty$.**

If

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty,$$

then

$$\lim_{x \to a} f(x)g(x) = ???$$

This is called <u>indeterminate of type $0 \cdot \infty$</u>.

Indeterminate products

Solution: If $\lim_{x \to a} f(x)g(x)$ is indeterminate of type $0 \cdot \infty$, then

- $\lim_{x \to a} \frac{f(x)}{\left(\frac{1}{g(x)}\right)}$ is indeterminate of type $\frac{0}{0}$.
- $\lim_{x \to a} \frac{g(x)}{\left(\frac{1}{f(x)}\right)}$ is indeterminate of type $\frac{\infty}{\infty}$.

Example: Compute $\lim_{x \to 0^+} x \log(x)$.

2. **Exponents of the form 0^0.** If

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x),$$

then

$$\lim_{x \to a} f(x)^{g(x)} = ???$$

This is called indeterminate of type 0^0.

Indeterminate exponents

Solution: If $\lim_{x \to a} f(x)^{g(x)}$ is indeterminate of type 0^0, then

$$\lim_{x \to a} \log\left(f(x)^{g(x)}\right) = \lim_{x \to a} g(x) \log\left(f(x)\right)$$

is indeterminate of type $0 \cdot \infty$

Other indeterminate forms

3. **Exponents of the form 1^∞ and ∞^0.**

These are dealt with similarly to those of form 0^0.

4. **Limits of the form $\infty - \infty$.**

There is no special trick to these.

Examples

1. Compute $\lim_{x\to\infty}(1+x)^{1/x}$.

2. Compute $\lim_{x\to 0^+}(\tan(x))^x$.

3. Compute $\lim_{x\to 0^+}\csc(x)-\cot(x)$.

4. Compute $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\log(x)}\right)$.

Concavity

We saw that a function is said to increase or decrease if its values literally go up or go down on an interval. We also saw that if a function is differentiable, we can use the sign of its derivative to determine whether it increases or decreases.

We'll now study what happens when the *derivative* of a function increases or decreases, and see that we can use the sign of its *second* derivative to determine this, and how it affects the shape of its graph.

Definition

Let f be a function and let I be an open interval on which it is differentiable.

We say that f is <u>concave up on I</u> if f' is increasing on I.

We say that f is <u>concave down on I</u> if f' is decreasing on I.

Inflection points

Analogously to local maxima/minima, we give names to the places where concavity changes.

Definition

Let f be continuous at a point a and differentiable near (but not necessarily at) a.

The point $(a, f(a))$ is called an inflection point if $\exists \delta > 0$ such that f is concave up (resp. concave down) on $(a - \delta, a)$ and concave down (resp. concave up) on $(a, a + \delta)$.

Example: Find the intervals on which $f(x) = \tan(x)$ is concave up and concave down, and find any inflection points.

Concavity and second derivatives

If f is a differentiable function, then f' is also a function. As we know already, we can determine if a differentiable function increases or decreases by examining its derivative.

So we should expect that if f' is differentiable (or in order words if f is twice differentiable), we can determine if f is concave up or concave down by examining f''.

Theorem

Let f be twice differentiable on an open interval I. Then:

- *If $f''(x) > 0$ for all $x \in I$, then f' increases on I, and therefore f is concave up on I.*
- *If $f''(x) < 0$ for all $x \in I$, then f' decreases on I, and therefore f is concave down on I.*

Proof: Exercise.

Concavity and second derivatives

Using the previous result, we can apply the local extreme value theorem to this situation to say something about what happens at an inflection point.

Theorem

Let f be differentiable on an open interval I, and suppose $(a, f(a))$ is an inflection point for some $a \in I$.

Then f' has a local maximum or local minimum at a, and therefore either $f''(a) = 0$, or f is not twice differentiable at a.

Proof: Exercise.

NOTE: The converse is not true!

Example

Let $f(x) = \dfrac{x}{x^2+1}$.

On which intervals is f increasing or decreasing?
On which intervals is f concave up or concave down?
Sketch its graph.

To save you time:

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

Asymptotes

We basically know all the necessary theory to talk about asymptotes. We just have to give names to some things so they're clearly defined.

Definition

The line $x = a$ is called a vertical asymptote of a function f if at least one of the following is true:

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty.$$

In particular, note that we only need an infinite limit on *one side* of a.

Caution

Note that if f is a function such that $f(x)$ is a fraction, it's *not enough* to find places where the denominator equals zero. You must check that one of the side limits is infinite.

For example the function

$$f(x) = \frac{x^3 + x^2 - 2x + 8}{x^2 - 4}$$

has a vertical asymptote at $x = -2$ but not at $x = 2$.

Horizontal asymptotes

Definition

If M is a real number, the line $y = M$ is called a horizontal asymptote of a function f if at least one of the following is true:

$$\lim_{x\to\infty} f(x) = M \quad \text{or} \quad \lim_{x\to-\infty} f(x) = M.$$

Caution

Note, there's nothing in either of the previous definitions about "getting closer and closer to the line but never touching it".

A function can be defined at a vertical asymptote, or can cross a horizontal asymptote many times.

For example, we know that

$$\lim_{x \to \infty} \frac{\sin(x)}{x} = 0,$$

and therefore the line $y = 0$ is a horizontal asymptote of this function. But the function crosses this line infinitely many times.

Generalizing

It's easy to see that the limits in the previous definition can also be written in this form:

$$\lim_{x \to \pm\infty} \Big[f(x) - M \Big] = 0.$$

There's no reason to restrict ourselves to talking about functions approaching horizontal lines though.

Given another function g, we can say that <u>f behaves asymptotically like g</u> if

$$\lim_{x \to \pm\infty} \Big[f(x) - g(x) \Big] = 0.$$

In the special case where g is a non-horizontal (or *oblique*) line, g is called an <u>oblique</u> or <u>slant asymptote</u>. We'll see some examples of this later.

Curve sketching

We are now ready to sketch curves. Note that the important skill we are practising here is combining a bunch of data into one graph that satisfies all the required properties.

To this end, we'll start by giving you all the properties, and asking you to sketch a graph that satisfies them.

Example

Sketch the graph of a function f that satisfies all the following properties.

- f is a polynomial with odd degree and positive leading coefficient.
- f is odd.
- $f(3) = 0$.
- $f(-1) = 2$.
- $f'(x) = 0$ only when $x = -1,\ 0,$ or 1.
- f is increasing on $(-\infty, -1]$ and $[1, \infty)$, and decreasing on $[-1, 1]$.
- $f''(x) = 0$ only when $x = -0.5,\ 0,$ or 0.5.
- f is concave up on $(-0.5, 0)$ and $(0.5, \infty)$, and concave down on all other intervals on which it is defined.

Example

Sketch the graph of a function f that satisfies all the following properties.

- f is defined for all real numbers *except* ± 3, and so is f'.
- f is odd.
- f only crosses either axis once.
- $f(4) = 7$, and $f'(0) = f''(0) = 0$.
- f has vertical asymptotes at $x = \pm 3$, and all four one-sided limits are ∞ or $-\infty$.
- f has $y = x$ as an oblique asymptote
- f is increasing on $(-\infty, -4]$ and $[4, \infty)$, and decreasing on all other intervals on which it is defined.
- f is concave up on $(-3, 0)$ and $(3, \infty)$, and concave down on all other intervals on which it is defined.
- $f(x) > x$ for all sufficiently large values of x.

Example

Sketch the function $f(x) = \dfrac{x-1}{\sqrt{4x^2-1}}$.

$$f'(x) = \frac{4x-1}{(4x^2-1)^{3/2}}.$$

$$f''(x) = -\frac{4(8x^2-3x+1)}{(4x^2-1)^{5/2}}.$$

Example

A weirder one:

Sketch the function $f(x) = xe^{1/x}$.

$$f'(x) = \frac{e^{1/x}(x-1)}{x}$$

$$f''(x) = \frac{e^{1/x}}{x^3}$$