MAT137 - Week 9

- Your third problem set is due **today, at 11:59pm**. Don't leave the submission process until the last minute.
- Today's lecture is about related rates problems, the local EVT, and Rolle's Theorem.
- Next week is reading week!
- For the following week's lecture, watch **all remaining videos in Playlist 5.**

Exercises on inverses

Last week I gave you these two exercises to do at home.

Problem 1. Suppose f is a function that is differentiable and injective on all of $\mathbb{R}$.

What would you need to be true in order for f^{-1} to have a vertical tangent line at $x = 7$? Sketch the graph of such a function.

Problem 2. Sketch the graph of $y = g(x)$ for a function g that satisfies **all** of the following properties:

- The domain of g is $\mathbb{R}$.
- g is continuous everywhere except at -2.
- g is differentiable everywhere except at -2 and 1.
- g is injective (on its entire domain).
- $(g^{-1})'(-4) = 2$.

Related rates

Idea of these problems: If you know a relationship between two quantities, you can derive a relationship between rates of change of those two quantities.

For example: If you know how the area A of a circle relates to its radius R ($A = \pi R^2$), and you know the area is changing at some rate $\frac{dA}{dt}$, then you can figure out the rate $\frac{dr}{dt}$ at which the radius must be changing.

You can do this by differentiating both sides of our relationship with respect to time:

$$A = \pi R^2 \implies \frac{d}{dt}[A] = \frac{d}{dt}\left[\pi R^2\right] \implies \frac{dA}{dt} = 2\pi R \frac{dR}{dt}$$

Exercise

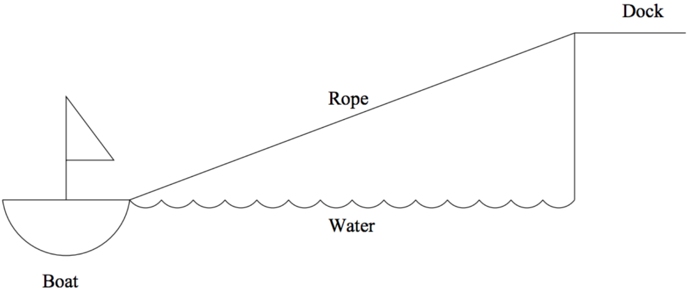
Here are two classic related rates problem, to start us off.

Problem 1. A 10 foot ladder leans against a wall. The bottom of the ladder starts slipping away at a rate of 0.5 feet per second. How quickly is the top of the ladder dropping when the bottom is 4 feet from the wall?

Problem 2. A spherical balloon is being inflated with 1 cubic metre of air per hour. How quickly is its diameter increasing when it is 2 metres in diameter?

Exercise (WE DID NOT SEE THIS ONE DURING CLASS)

Problem 3. A boat is being drawn towards a dock by pulling a rope at a constant rate.

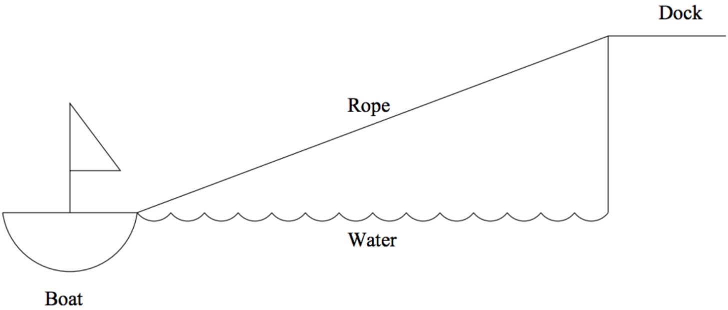


How does the rate at which the rope is being pulled in relate to the rate at which the boat approaches the dock?

1. They are equal.
2. One is a constant multiple of the other.
3. Neither of the above.

Exercise (WE DID NOT SEE THIS ONE DURING CLASS)

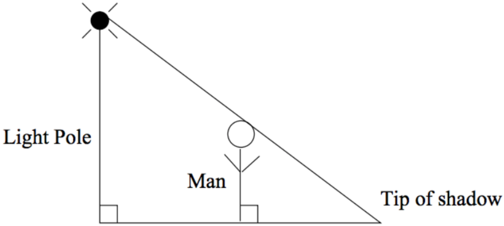
Problem 4. Same situation:



True or **false**: The closer the boat gets to the dock, the faster it moves.

Exercise

Problem 5. A man walks away from a lamp post at a constant rate.

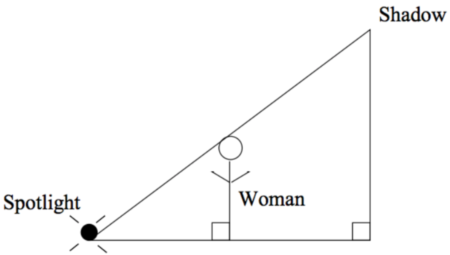


How does the rate at which he walks relate to the rate at which the length of his shadow grows?

1. They are equal.
2. One is a constant multiple of the other.
3. Neither of the above.

Exercise

Problem 6. A woman is between a splotlight on the ground and a wall, and walks toward the wall at a constant rate.



How does the rate at which he walks relate to the rate at which the length of her shadow changes?

1. They are equal.
2. One is a constant multiple of the other.
3. Neither of the above.

Extreme values

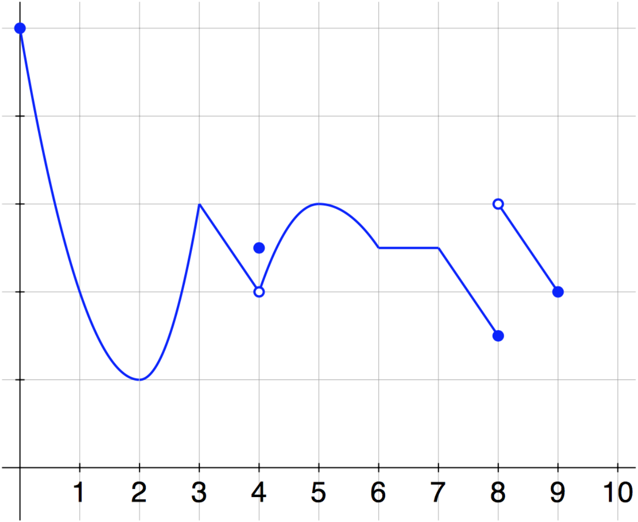
Let f be a function defined on a domain D. Recall the following definitions:

- Let $c \in D$. f has a <u>maximum</u> at c if $f(x) \leq f(c)$ for all $x \in D$.
- Let $c \in D$. f has a <u>minimum</u> at c if $f(x) \geq f(c)$ for all $x \in D$.
- Let $c \in D$. f has a <u>local maximum</u> at c if $\exists \delta > 0$ such that $|x - c| < \delta \implies f(x) \leq f(c)$.
- Let $c \in D$. f has a <u>local minimum</u> at c if $\exists \delta > 0$ such that $|x - c| < \delta \implies f(x) \geq f(c)$.

Maxima and minima are also often called <u>global extrema</u>.

Extreme values

Find all local and global extrema of this function.



Local EVT

I didn't show this slide during lecture, but I'm including it for completeness.

Recall this theorem from one of the videos.

Theorem (Local EVT, or sometimes Fermat's Theorem)

Let f be a function defined on an interval I, and suppose f has a local maximum or local minimum at an interior point c of I.

Then either $f'(c) = 0$, or f is not differentiable at c.

That leads us to the following definition:

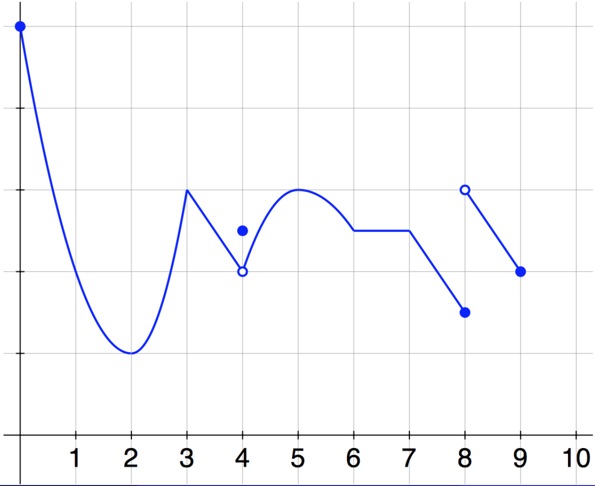
Definition

Let f be defined on an interval I. An interior point c of I is called a <u>critical point</u> if $f'(c) = 0$ or f is not differentiable at c.

Critical points

I didn't show this slide during lecture, but I'm including it for completeness.

Here's that function again. Find all the critical points.



Rolle's Theorem

Theorem (Rolle's Theorem)

Let f be a function such that:

- *f is continuous on an interval $[a, b]$,*
- *f is differentiable on the corresponding open interval (a, b),*
- *$f(a) = f(b)$.*

Then, there is a $c \in (a, b)$ such that $f'(c) = 0$.

Make sure to draw yourself a picture of what this theorem says.

Application of Rolle's Theorem

The main reason we mention Rolle's theorem is as a lead up to the Mean Value Theorem. But Rolle's theorem alone has a number of applications.

We'll discuss two of them. First:

Theorem

Let f be differentiable on $\mathbb{R}$ and suppose $f'(x) > 0$ for all $x \in \mathbb{R}$.

Then f has at most one zero.

Problem. Prove this theorem.

Hint: What would happen if f had more than one root?

Application of Rolle's Theorem

Another very popular application of Rolle's Theorem, which is almost exactly the same as the previous result.

Theorem

If f is differentiable on $\mathbb{R}$ and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is injective.

Problem. Prove this theorem.

Hint: There is more than one way to prove this.

You can try a similar strategy to the previous proof, and ask what would happen if f was not injective.

Or you can try to prove it as a consequence of the previous theorem.

Exercise

Problem. Consider the function f defined by:

$$f(x) = e^x - \sin x + x^2 + 10x.$$

How many zeroes does this function have?