MAT137 - Week 8

- Your third problem set is due **Thursday, 2 November**.
- Today's lecture is about inverse functions (including inverse trig functions).
- For next week's lecture, watch the first six videos on Playlist 5.
- You have two homework problems on the last slide.

More logarithmic differentiation

We'll warm up today by doing some differentiation.

Problem 1. Let f be the function defined by $f(x) = \log_{(x^2)}(x+1)$.

That is, $f(x)$ is the base-x^2 logarithm of $x+1$.

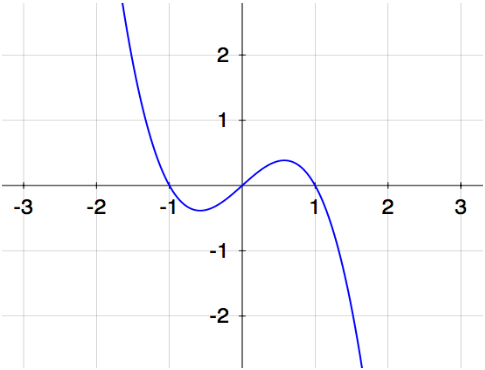
Compute $f'(x)$.

Problem 2. Consider the equation $y^x + x^y = x$. Compute $\frac{dy}{dx}$.

(This isn't particularly interesting, but it shows how you can now differentiate most things you write down.)

Exercise

Let f be the following function:



1. What is the largest interval containing -1 on which f has an inverse?
2. What is the largest interval containing 0 on which f has an inverse?

Sketch the graphs of these two inverses.

The arcsin function

In one of the videos, we defined the function arcsin as the inverse of the function

$$g(x) = \sin x, \text{ restricted to the interval } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

In other words,

$$\arcsin(x) = \theta \iff \begin{cases} \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \sin\theta = x \end{cases}$$

The arcsin function

Remember that the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is simply a choice we make in order to obtain an injective function from sin.

This is just like how we choose to define $\sqrt{x}$ as the positive number whose square is x.

We can restricte $\sin x$ to another interval, like

$$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \text{ or } \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right].$$

and define an inverse for those restrictions as well.

Those would be perfectly well-defined functions, but none of them would be arcsin.

Doing computations with arcsin

Using this definition from above...

$$\arcsin(x) = \theta \quad \iff \quad \begin{cases} \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \sin\theta = x \end{cases}$$

...compute the following:

1. $\sin(\arcsin(\frac{1}{2}))$, $\cos(\arcsin(\frac{1}{2}))$, $\tan(\arcsin(\frac{1}{2}))$.
   (Do these three *without computing* $\arcsin(\frac{1}{2})$.)
2. $\sin(\arcsin(2))$
3. $\arcsin(\sin(1))$
4. $\arcsin(\sin(7))$ (Hint: The answer is not 7.)
5. $\arcsin(\sin(6))$

The derivative of $\arcsin(x)$

Recall that to compute $\frac{d}{dx}\arcsin(x)$, we use implicit differentiation.

By definition, we know that

$$\sin(\arcsin(x)) = x$$

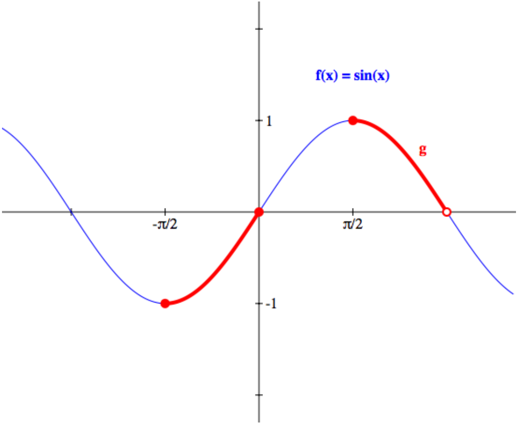
for all $x \in [-1, 1]$.

Differentiate both sides of this expression with respect to x. Then rearrange to obtain:

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1 - x^2}}.$$

A different restriction of sin

Consider the following restriction g (in red) of the graph of $f(x) = \sin x$:

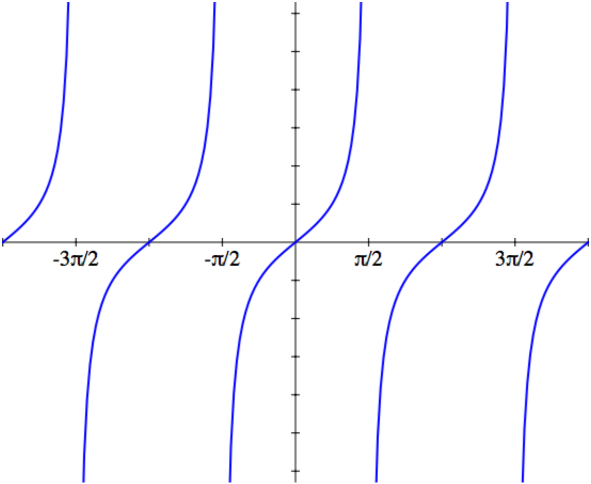


Problem 1. Does g have an inverse? If so, what are its domain and range?

Problem 2. Sketch the graph of g^{-1}. What do you notice about it?

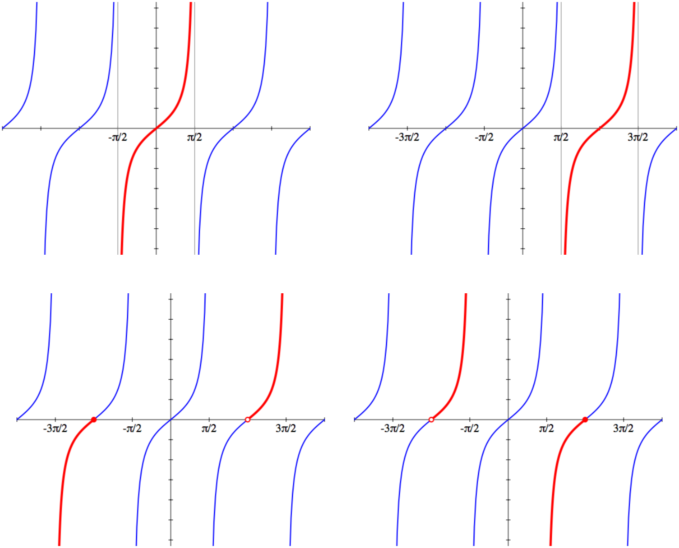
The arctan function

Here's the graph of (part of) the tangent function.



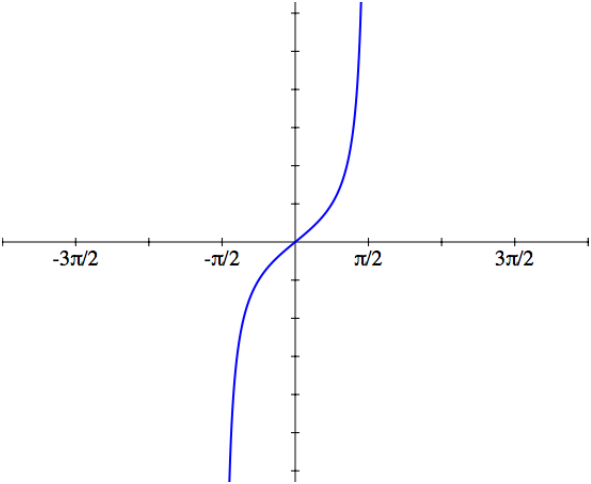
Does this function have an inverse?

No! We have to restrict it. Any of the following would do:



The arctan function

By convention, we use the first one. That is, we define arctan to be the inverse of the function with this graph:



The arctan function

In symbols, that means we define the function arctan as the inverse of the function

$$g(x) = \tan x, \text{ restricted to the interval } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

In other words,

$$\arctan(x) = \theta \iff \begin{cases} \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \tan\theta = x \end{cases}$$

The arctan function

Here's the definition of arctan again:

$$\arctan(x) = \theta \iff \begin{cases} \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \tan\theta = x \end{cases}$$

Problem 1. What are the domain and range of arctan?

Problem 2. Sketch the graph of arctan.

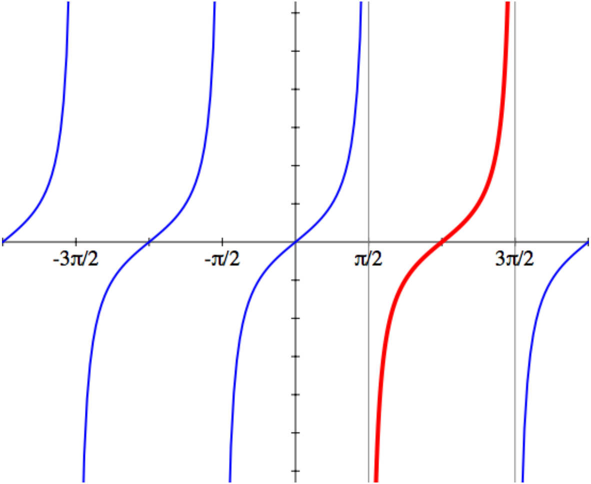
Problem 3. Compute the following values:

1. $\tan(\arctan(12))$.
2. $\arctan(\tan(0))$.
3. $\arctan(\tan(\pi))$.
4. $\arctan(\tan(7))$.

Problem 4. Compute the derivative of arctan.

A different inverse for tan

We said that any of a number of restrictions of tan would be injective, so let's see what happens with one of them. Let's use this one:



A different inverse for tan

We define the function g as the inverse of the function

$$f(x) = \tan x, \text{ restricted to the interval } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

In other words,

$$g(x) = \theta \iff \begin{cases} \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ \tan \theta = x \end{cases}$$

A different inverse for tan

Here's the definition of g again:

$$g(x) = \theta \quad \iff \quad \begin{cases} \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ \tan\theta = x \end{cases}$$

Problem 1. What are the domain and range of g?

Problem 2. Sketch the graph of g.

Problem 3. Compute the following values (and compare your answers to the ones you obtained for arctan):

1. $\tan(g(12))$.
2. $g(\tan(0))$
3. $g(\tan(\pi))$.
4. $g(\tan(7))$.

Problem 4. Compute the derivative of g.

Exercises on inverses

HOMEWORK PROBLEMS

Problem 1. Suppose f is a function that is differentiable and injective on all of $\mathbb{R}$.

What would you need to be true in order for f^{-1} to have a vertical tangent line at $x = 7$? Sketch the graph of such a function.

Problem 2. Sketch the graph of $y = g(x)$ for a function g that satisfies **all** of the following properties:

- The domain of g is $\mathbb{R}$.
- g is continuous everywhere except at -2.
- g is differentiable everywhere except at -2 and 1.
- g is injective (on its entire domain).
- $(g^{-1})'(-4) = 2$.