MAT137 - Week 7

- Your first midterm test is **tomorrow**. See the course website for details about the rooms and stuff.
- Today's lecture is about derivatives.
- For next week's lecture, start watching the first few videos on Playlist 4, and check my webpage on the weekend for the exact videos you'll need for next class.

Extending the power rule.

In the videos you saw a proof of the power rule for natural exponents, done by induction:

Theorem

Let n be a positive integer, and let f be the function defined by $f(x) = x^n$.

Then f is differentiable everywhere, and

$$f'(x) = nx^{n-1}.$$

Extending the power rule.

Now that we know the quotient rule, we can extend this result to negative exponents.

Prove the following theorem:

Theorem

Let n be a positive integer, and let f be the function defined by $f(x) = x^{-n}$.

Then f is differentiable everywhere except zero, and

$$f'(x) = -nx^{-n-1} = -\frac{n}{x^{n+1}} \quad (x \neq 0)$$

What would need in order to extend this to rational numbers? In other words, if a and b are integers, what would we have to know in order to compute the derivative of $f(x) = x^{a/b}$?

Extending the power rule.

If $y = x^{a/b}$, then what tools that we know now can we use to compute $\frac{dy}{dx}$?

Well, we know that $y^b = x^a$. If we could differentiate this expresion we might get somewhere.

We'll come back to this later.

Problem. Let $f(x) = \dfrac{1}{x^7}$.

1. Calculate the first few derivatives of f.

2. Make a conjecture for a formula for the n^{th} derivative of f.

3. Prove your formula.

True or false?

Problem. Suppose f and g are differentiable everywhere. Is the following equation true or false?

$$(g \circ f)'(x) = g'(f(x)).$$

False! You already know this even with what little we know about derivatives.

For example if $f(x) = g(x) = x^2$, then $(g \circ f)(x) = (x^2)^2 = x^4$ and so $(g \circ f)'(x) = 4x^3$.

But $g'(x) = 2x$, and so $g'(f(x)) = 2f(x) = 2x^2$.

The Chain Rule

The solution to this is the chain rule:

Theorem

Let $a \in \mathbb{R}$, and let f and g be functions.

If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Using the Chain Rule

Problem. Compute the derivatives of the following functions:

1. $f(x) = (3x^7 + 4x + 1)^{2017}$.

2. $g(x) = (x^2 + 1)^7(4x^3 + 2x)^{13}$.

Note, these are derivatives you technically could have computed before. The Chain Rule just makes things much easier.

Chain Rule + Product Rule $\implies$ Quotient Rule

Problem 1. Let $a \in \mathbb{R}$, and let g be a function which is differentiable at a and such that $g(a) \neq 0$.

Let $h(x) = \dfrac{1}{g(x)}$.

Use the chain rule to derive a formula for $h'(x)$.

Problem 2. Use your formula from above to give a simple proof of the quotient rule.

To this day, this is how I remember the quotient rule.

Trigonometric derivatives

In one of the videos, you learned that:

$$\frac{d}{dx} \sin x = \cos x \quad \text{and} \quad \frac{d}{dx} \cos x = -\sin x$$

Problem 1. Evaluate the following limit:

$$\lim_{h \to 0} \frac{\sin(7x + h) - \sin(7x)}{h}.$$

Problem 2. Evaluate the following limit:

$$\lim_{h \to 0} \frac{\cos(7(x + h)) - \cos(7x)}{h}.$$

In one of the videos, you learned that

$$\frac{d}{dx}\, e^x = e^x.$$

And in general, if a is a positive real number, then

$$\frac{d}{dx}\, a^x = \log(a)\, a^x.$$

Note: I always denote natural logarithms with just "log". No one in math cares about base 10 logarithms...

Using these formulas.

Problem. Compute the derivatives of the following functions:

1. $f(x) = \tan(x) + e^{\sin x}$.

2. $g(x) = \log(7^{\sin x} + 1)$.

3. $h(x) = \sin(\cos(\log(7^x)))$

Explicit functions

An equation like $y = x^2 + \sin(x)$ expresses a relationship between values of x and y. More specifically, it says that y an explicit function of x. The function is $f(x) = x^2 + \sin(x)$.

If we want to figure out how y varies when x varies, we can simply differentiate f, in this case getting

$$\frac{dy}{dx} = 2x + \cos(x).$$

Implicit functions

The equation $x^2 + y^2 = 1$ also expresses a relationship between values of x and y. In this case though, the values of y cannot be expressed as an explicit function of x.

That is, there is no function f such that the equation $y = f(x)$ encapsulates all the information in the earlier equation.

But we still might want to ask how y varies when x varies.

Implicit functions

In the particular case of $x^2 + y^2 = 1$, we know that by splitting into two cases - when y is non-negative or non-positive - the relationship in each case can be expressed by an explicit function:

When $y \geq 0$, we know $y = \sqrt{1 - x^2}$.
When $y \leq 0$, we know $y = -\sqrt{1 - x^2}$.

Implicit functions

We can also differentiate both of these functions to find out how y varies when x varies:

When $y \geq 0$, we find $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1-x^2}} \left(= \dfrac{-x}{y}\right)$.
When $y \leq 0$, we find $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1-x^2}} \left(= \dfrac{-x}{-\sqrt{1-x^2}} = \dfrac{-x}{y}\right)$.

So, no matter what x is, it turns out that $\dfrac{dy}{dx} = \dfrac{-x}{y}$. This equation *accounts for both cases*.

Implicit differentiation

Instead of splitting up the cases, we could have done all of this at once by *implicitly differentiating* the original equation $x^2 + y^2 = 1$.

To do this you differentiate both sides of the equation, and treat y as though it's a function of x.

So for example if you see a y^2, you apply the Chain Rule:

$$\frac{d}{dx}\left(y^2\right) = 2y\, y'.$$

In this case you'd get:

$$2y\, y' + 2x = 0 \implies y' = -\frac{x}{y}.$$

Notice that the RHS of this formula doesn't make sense when $y = 0$. That makes sense, since *y cannot be thought of as a function of x* around those points.

Implicit differentiation

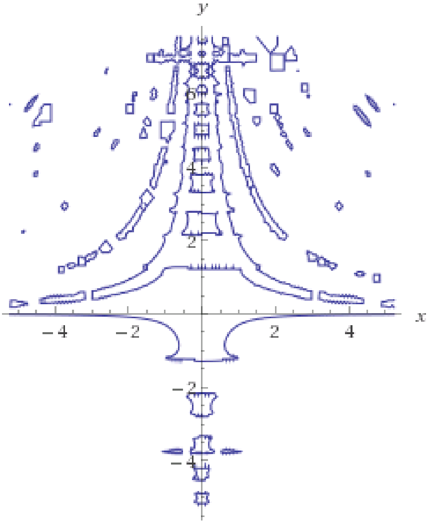
In general, given some complex relationship between x and y, like this...

$$\tan^2(xy) = 3x^2y + \sec(y^2)$$

...it's just impossible to split into cases in which you can express y as an explicit function of x, so implicit differentiation is our only tool that always works.

Implicit differentiation

By the way, here's what that curve looks like:



Using implicit differentiation to extend power rule

Implicit differentiation is useful even when we do have y defined as an explicit function of x.

Now that we know how to implicitly differentiate, we can finally extend the power rule to rational exponents.

Recall that we said that if $y = x^{a/b}$, then we can write this relationship as:

$$y^b = x^a.$$

Now we know how to compute y' for a relationship like this.

Problem. Compute y', and then express it entirely in terms of x.

Implicit differentiation

Problem. For the horrible curve from earlier:

$$\tan^2(xy) = 3x^2y + \sec(y^2)$$

...try to compute y'.

It's long, but simple.

True or false?

Problem. Let $f(x) = (x+1)^x$. Is the following formula true?

$$f'(x) = x \cdot (x+1)^{x-1}.$$

False! This formula is trying to use the power rule for a situation it can't be used for.

The power rule only applies to functions of the form $g(x) = x^{\text{constant}}$.

Logarithmic differentiation to the rescue!

Logarithmic differentiation.

The idea behind this technique is simple:

Logarithms make complicated expressions simpler.

We know the following rules for logarithms:

$$\log(ab) = \log(a) + \log(b) \quad \text{and} \quad \log(a^b) = b\log(a).$$

The first one of these is helpful because derivatives play much more nicely with sums than they do with products, but usually we just use the product rule.

The second one of these is helpful because at the moment we don't know how to compute the derivative of something like $f(x)^{g(x)}$.

Logarithmic differentiation.

Logarithms help us with problems like this because if you take a logarithm of a function defined like this, it gets simpler.

Problem. Compute the derivatives of the following functions:

1. $f(x) = (x+1)^x$.

2. $g(x) = x^x + [\cos(x)]^x$

3. Now try to generalize these ideas into a new differentiation rule:

Let f and g be differentiable functions, and define h by

$$h(x) = [f(x)]^{g(x)}.$$

Derive a formula for $h'(x)$.