MAT137 - Week 6

- Problem Set 2 is due **today**, by 11:59pm. Don't leave the submission process to the last minute!
- Your first midterm test is next Friday, 20 October.
  If you have a conflict with the test, the deadline to tell us about it is tomorrow! See the course website for details.
- Today's lecture is (mostly) about derivatives.
- For next week's lecture, watch all of the remaining videos on Playlist 3.

Homework from last class

Last class you were instructed to write a proof of this theorem for homework. We did most of the "rough work" in class and we looked at a bad "proof" together. Today we'll concentrate on writing the proof well.

Theorem

Let c be a real number, and let f and g be functions defined everywhere except possibly at c.

Assume:

- $\lim_{x \to c} f(x) = 0.$
- *g is bounded. That means:*

$$\exists M > 0 \text{ such that } \forall x \neq c, \ |g(x)| < M.$$

Then $\lim_{x \to c} [f(x)g(x)] = 0.$

Some things to check your proof for.

- Is the structure of the proof correct?
  (ie. Do you start by fixing an arbitrary ε, then choose a δ that depends only on ε, etc.)
- Did you precisely say what δ is?
- Is your proof self-contained? (ie. Does it reference rough work that isn't written in the proof?)
- Are all of your variables defined? In the right order?
  (In particular, remember that a variable that is quantified in a statement doesn't have a value outside that statement.)
- Does each step follow logically from the previous steps? Have you explained why?
- Do you make sure not to start by assuming the conclusion?

Problem 1. Is the following statement true or false?

> *Let f be a function. The tangent line to the graph of $y = f(x)$ at $x = c$ touches the curve only at the point $(c, f(c))$.*

Problem 2. Which of the following is true about the tangent line to the graph of $y = x$ at $x = 7$?

1. The tangent line is $y = 7$.
2. The tangent line is $y = x$.
3. There is no tangent line to this graph at $x = 7$.
4. There are infinitely many tangent lines to this graph at $x = 7$.

Derivatives.

Problem 3. Let $c \in \mathbb{R}$, and suppose f is a function defined everywhere except at c. Which of the following must be true?

1. f is differentiable at c.
2. f cannot be differentiable at c.
3. We cannot say. f may or may not be differentiable at c.

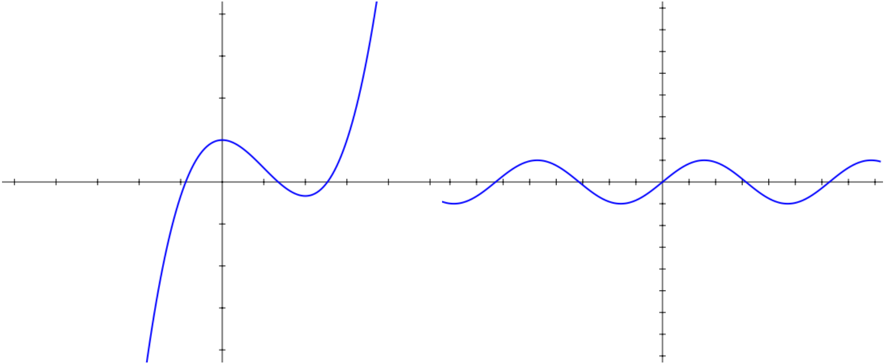
Problem 4. Let $c \in \mathbb{R}$, and suppose f is differentiable everywhere except at c. Also suppose that f' is continuous everywhere except at c. Which of the following must be true?

1. f is continuous at c.
2. f cannot be continuous at c.
3. We cannot say. f may or may not be continuous at c.

Problem 5. Give an example of a function that is continuous everywhere, but whose derivative is not continuous at exactly one point.

Shapes of graphs

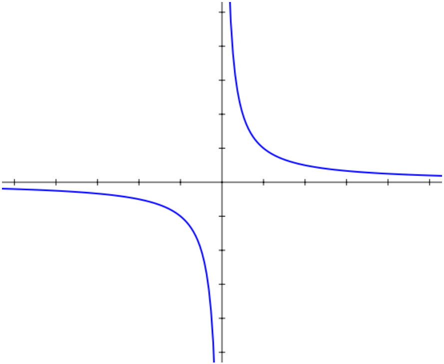
Sketch the graphs of the derivatives of these functions.

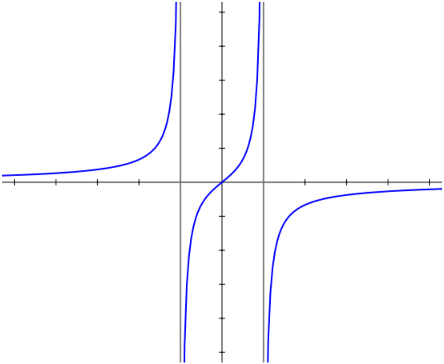


Shapes of graphs

For each of these two graphs:

1. Sketch the graph of a continuous function that goes through the origin and whose derivative looks like the graph.
2. Sketch the graph of a non-continuous function whose derivative looks like the graph.





Definition of a derivative.

Recall the definition of differentiability and of a derivative:

Definition

Let c be a real number, and let f be a function defined at least on an open interval containing c. f is said to be differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists.

In this case its value is denoted by $f'(c)$, and called the derivative of f at c.

Note that the two limits in there are equal via the change of variables $h = x - c$. Convince yourself of this.

We didn't do this problem in class, but I'm including it here anyway.

Problem 1. Let f be the following function:

$$f(x) = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

Is f differentiable at 0? If so, what is its derivative?

Another way to write this function is $f(x) = x|x|$. Convince yourself of this.

Problem 2. Let g be the following function:

$$g(x) = \begin{cases} \sin x & x < 0 \\ x & x \geq 0 \end{cases}$$

Is g differentiable at 0? If so, what is its derivative?

Proving the quotient rule.

Recall the quotient rule for derivatives from the videos, which I'll state formally here:

Theorem

Let $c \in \mathbb{R}$. Let f and g be functions defined at c and near c, and assume that $g(c) \neq 0$.

Define a function h by $h(x) = \dfrac{f(x)}{g(x)}$.

If f and g are differentiable at c, then h is differentiable at c, and

$$h'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}.$$

First, use the definition of $h'(c)$ to write down the limit you need to prove.

Then prove it.

Recall a trick from the product rule.

In order to prove the product rule, we had to prove a similar limit, and to do that we did a simple "trick" of adding zero in a creative way:

$$\begin{aligned}
&\frac{f(x)g(x) - f(c)g(c)}{x-c} \\
=&\frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x-c} \\
=&\frac{f(x)-f(c)}{x-c}\,g(x) + f(c)\frac{g(x)-g(c)}{x-c}
\end{aligned}$$

A similar trick will help you with this proof.

Be careful to explicitly justify any limits you evaluate in your proof. In many (but not all) cases that will involve ensuring the hypotheses of the Limit Laws are satisfied.