MAT137 - Week 5

- The last day to drop down to MAT135 without penalty is **tomorrow**.
- Today's lecture is about limits and continuity, still.
- For next week's lecture, start watching at least the first four videos on Playlist 3. Check my website on the weekend for exactly how many you should watch.
- Note: You have homework from this lecture. See slides 4 through 7.

A quick warm up exercise.

Suppose a is a real number, and suppose f is a function that is not defined at a.

Which of the following statements must be true?

- $\lim_{x \to a} f(x)$ exists.
- $\lim_{x \to a} f(x)$ doesn't exist.
- We can't conclude anything. $\lim_{x \to a} f(x)$ may or may not exist.

Which of the following statements must be true?

- f is continuous at a.
- f is not continuous at a.
- We can't conclude anything. f may or may not be continuous at a.

Is this theorem true?

Theorem

Let c be a real number, and let f and g be functions defined everywhere except possibly at c.

If $\lim_{x \to c} f(x) = 0$, then $\lim_{x \to c} [f(x)g(x)] = 0$.

If you think it's not true, give a counterexample.

Let's prove a new theorem.

Theorem

Let c be a real number, and let f and g be functions defined everywhere except possibly at c.
Assume:

- $\lim_{x \to c} f(x) = 0$.
- *g is bounded. That means:*

$$\exists M > 0 \text{ such that } \forall x \neq c,\ |g(x)| < M.$$

Then $\lim_{x \to c} [f(x)g(x)] = 0$.

- Write down the formal definition of what you have to prove.
- Before thinking about anything, write down the structure of the proof.
- Do some rough work to figure out how your assumptions relate to what you need to do.
- Then, write the proof.

Some things your proof should do.

- Is the structure of the proof correct?
  (ie. Do you start by fixing an arbitrary ε, then choose a δ that depends only on ε, etc.)
- Did you precisely say what δ is?
- Is your proof self-contained? (ie. Does it reference rough work that isn't written in the proof?)
- Are all of your variables defined? In the right order?
- Does each step follow logically from the previous steps? Have you explained why?
- Do you make sure not to start by assuming the conclusion?

What's wrong with this "proof"?

Proof.

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - c| < \delta \implies |f(x)g(x)| < \varepsilon.$$

$$\forall \varepsilon_1 > 0, \exists \delta_1 > 0 \text{ such that } 0 < |x - c| < \delta_1 \implies |f(x)| < \varepsilon_1.$$

$$\exists M > 0 \text{ such that } \forall x \neq c, \ |g(x)| < M.$$

$$|f(x)g(x)| = |f(x)||g(x)| < \varepsilon_1 M.$$

$$\varepsilon = \varepsilon_1 M \implies \varepsilon_1 = \frac{\varepsilon}{M}.$$

$$\text{Therefore } \delta = \delta_1.$$

□

Most of the right ideas are here, but this is not a proof. Make sure your proof doesn't have all of these problems.

Homework!

Write a proof of the theorem as homework.

At what points is this function continuous?

$$f(x) = \begin{cases} \dfrac{x^2 - 49}{x - 7} & x \neq 7 \\ 0 & x = 7 \end{cases}$$

It's continuous everywhere except at 7.

Can you redefine $f(7)$ so that f is continuous everywhere?

At which points is this function continuous?

Recall that given a real number x, the floor of x, denoted by $\lfloor x \rfloor$, is the largest integer that's smaller than or equal to x.

For example, $\lfloor 7 \rfloor = 7$, $\lfloor \pi \rfloor = 3$, and $\lfloor -4.5 \rfloor = -5$.

At which points (if any) in the interval $(-2\pi, 2\pi)$ the following function discontinuous?

$$f(x) = \lfloor \sin x \rfloor.$$

Sketch the graph of this function.

Limits and compositions

Let $c \in \mathbb{R}$, and let f and g be functions that are defined on all of $\mathbb{R}$.

Assume that $\lim\limits_{x \to c} g(x)$ exists and equals a real number L.

Is it necessarily true that

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)?$$

No! We need f to be continuous at L.

Come up with a counterexample. (Hint: You've already seen several of them.)

Limits and compositions

The theorem from the videos about this:

Theorem

Let $c \in \mathbb{R}$, and let f and g be functions such that:

- $\lim_{x \to c} g(x) = L$.
- *f is continuous at L.*

Then,

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

Limits and compositions.

Problem. Find functions f and g, defined on all of $\mathbb{R}$, such that:

- f and g are both continuous at 0.
- $f(g(x))$ is not continuous at $x = 0$.

Does this contradict the theorem we just saw? Why or why not?

The Intermediate Value theorem

Theorem

Let $a < b$ be real numbers, and let f be a function defined on the interval $[a, b]$.

If f is continuous on $[a, b]$, then f takes all values between $f(a)$ and $f(b)$.

Another way of saying this (my preferred way):

Theorem

If f is a continuous function, then f sends intervals to intervals.

That is, if A is an interval contained in the domain of f, then:

$$f(A) = \{\, f(x) \,:\, x \in A \,\} \text{ is an interval.}$$

The Extreme Value theorem

Theorem

Let $a < b$ be real numbers, and let f be a function defined on the interval $[a, b]$.

If f is continuous on $[a, b]$, then f takes on a maximum and a minimum value on $[a, b]$.

Using these theorems

Problem 1. Use the IVT to prove that you were 3 feet tall at some point in your life.

Problem 2. Does the EVT imply that you will attain a maximum height?

(This turned into a deep philosophical discussion with my office mate earlier today...)

Problem 3. Suppose that at half time during a Raptors basketball came, the Raptors have 51 points.

Is it necessarily true that at some point during the first half, they had exactly 20 points?

Using these theorems

Problem 4. Prove that some point in Ivan's life, his height in inches equalled his weight in pounds.

Some data about Ivan:

- Height at birth: 20 inches.
- Weight at birth: 8 pounds.
- Height now: about 5 feet 9 inches.
- Weight now: None of your business!