MAT137 - Week 4

- Problem Set 1 is due **today** by 11:59pm. **Do not leave the submission process to the last minute.**
- Today's lecture is about limits, and a bit on continuity.
- For next week's lecture, watch the rest of the videos in Playlist 2.

The precise definition of a limit

Here's the precise definition of a limit given in the videos.

Definition

Let f be a function defined on an open interval containing a real number c, except possibly at c. Let L be a real number. Then

$$\lim_{x \to c} f(x) = L$$

means

$$\forall \varepsilon > 0 \, \exists \delta > 0 \text{ such that } 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

The precise definition of a limit

$$\forall\varepsilon > 0\,\exists\delta > 0 \text{ such that } 0 < |x - c| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

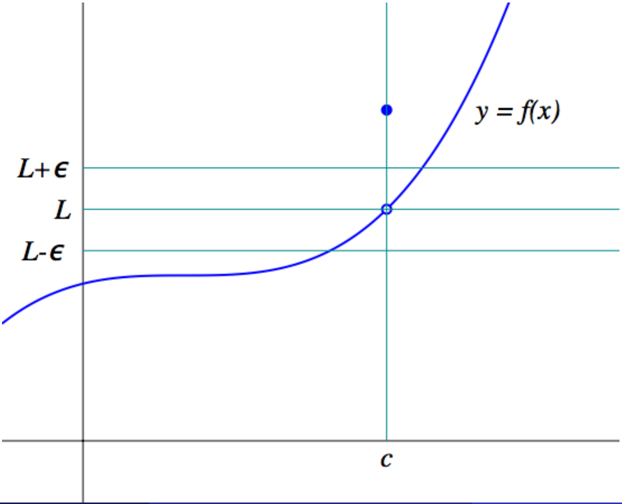
Translation into normal English:

$\forall\varepsilon > 0$	"No matter how close you require f to get, ..."
$\exists\delta > 0$ such that	"...there is a distance δ such that..."
$0 < \|x - c\| < \delta \Longrightarrow$	"...if x is within δ of (but not equal to) c, then..."
$\|f(x) - L\| < \varepsilon$	"...$f(x)$ is closer to L than you required."

In pictures

"No matter how close you require f to get, ..."

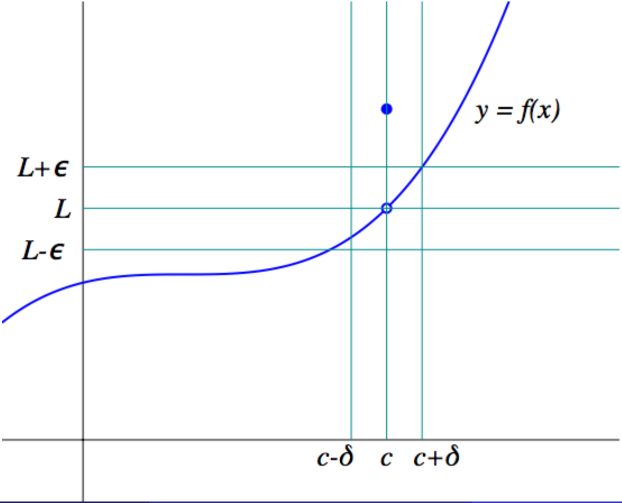
ie. No matter how thin this horizontal bar centred at L is...



In pictures

"...there is a distance δ..."

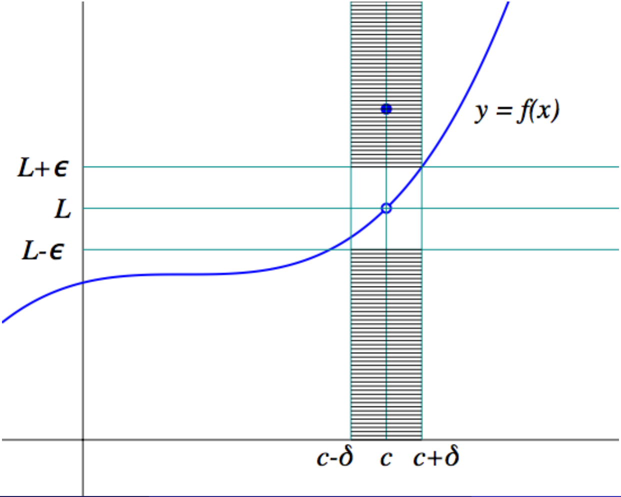
ie. ...there is a vertical bar centred at c...



In pictures

"...such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$."

ie. ...such that the function avoids the two shaded regions:



We did one of these proofs last class

Last class, you essentially proved that $\lim_{x \to 3} 4x = 12$.

Recall that we showed that for a fixed positive real number ε, the following conditional is true:

$$\text{If } |x - 3| < \frac{\varepsilon}{4}, \text{ then } |4x - 12| < \varepsilon.$$

So $\delta = \frac{\varepsilon}{4}$ works in the definition of this limit.

Proving limits exist

That value of δ was pretty easy to find, because we could just do this:

$$|4x - 12| = 4|x - 3|.$$

Once we saw that, it was obvious that for any positive ε:

$$|x - 3| < \frac{\varepsilon}{4} \implies |4x - 12| = 4|x - 3| < 4\frac{\varepsilon}{4} = \varepsilon.$$

This sort of situation is always the goal with $\varepsilon - \delta$ proofs.

Proving limits exist

Back to the general definition:

$$\forall \varepsilon > 0 \, \exists \delta > 0 \text{ such that } 0 < |x - c| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

Suppose we can find some positive constant D such that

$$|f(x) - L| \leq D|x - c|.$$

Then for any positive ε, we can do the same sort of thing as before:

$$|x - c| < \frac{\varepsilon}{D} \implies |f(x) - L| \leq D|x - c| < D\frac{\varepsilon}{D} = \varepsilon$$

In other words, if you can find such a D, then for any positive ε, setting $\delta = \dfrac{\varepsilon}{D}$ (or any smaller number, of course) will satisfy the definition.

Let's actually do it

Problem. Prove that $\lim_{x \to -1} 2x + 1 = -1$

Some intermediary things to do:

- Write out the $\varepsilon - \delta$ definition of this limit. In other words, write down what you have to prove.
- Look at the $|f(x) - L|$ term, and try to prove that it's less than or equal to $D|x - c|$, for some value of D.
- Figure out a δ that will work.
- Write the proof.

What's wrong with this proof?

Proof.

$$\begin{aligned} |(2x+1)-1| &< \varepsilon \\ |2x+2| &< \varepsilon \\ 2|x+1| &< \varepsilon \\ |x+1| &< \frac{\varepsilon}{2} \end{aligned}$$

So $\delta = \frac{\varepsilon}{2}$. $\square$

Nearly everything is wrong with this proof.

Sample proof

Proof.

We want to show that

$$\forall \varepsilon > 0\, \exists \delta > 0 \text{ such that } 0 < |x+1| < \delta \implies |(2x+1) - (-1)| < \varepsilon.$$

So, fix an $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{2}$, and assume that $0 < |x+1| < \delta$. Then:

$$|(2x+1) - (-1)| = |2x+2| = 2|x+1| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $|(2x+1) - (-1)| < \varepsilon$, as required. $\square$

Note that the structure of this proof was dictated by the structure of the $\varepsilon - \delta$ definition of the limit. All we had to fill in were the value of δ, and the little bit of math at the end.

A trickier example

We didn't do this one in lecture, but I'm providing it here.

Problem. Prove that $\lim_{x \to 1} x^2 = 1$

So we want to show that

$$\forall \varepsilon > 0 \, \exists \delta > 0 \text{ such that } 0 < |x - 1| < \delta \implies |x^2 - 1| < \varepsilon.$$

How can we relate $|x^2 - 1|$ to $|x - 1|$? Manipulate the former expression, and try to find $|x - 1|$ in it somewhere.

Difference of squares! $|x^2 - 1| = |x + 1||x - 1|$.

Good! We are free to force $|x - 1|$ to be small as we want, because we can choose δ. Would it make sense to begin the proof in the following way?

$$\text{Fix } \varepsilon > 0. \text{ Let } \delta = \frac{\varepsilon}{|x + 1|} \ldots$$

A trickier example

No, this is nonsense; the x in that expression is undefined/unquantified.

δ must be a single, positive number that can depend on ε, and nothing else.

Back to this expression: $|x^2 - 1| = |x+1||x-1|$.

We need to get control over the $|x+1|$ term somehow. If x is allowed to vary freely, $|x+1|$ can get really big. This seems bad.

But, you have the power to make x as close as you want to 1.

So put a limit on how far x can get from 1, and use that to put an upper bound on $|x+1|$.

Another tricky example

Problem. Prove that $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$

So we want to show that

$$\forall \varepsilon > 0\, \exists \delta > 0 \text{ such that } 0 < |x-2| < \delta \implies \left| \frac{1}{x} - \frac{1}{2} \right| < \varepsilon.$$

How can we relate $\left| \frac{1}{x} - \frac{1}{2} \right|$ to $|x-2|$? Manipulate the former expression, and try to find $|x-2|$ in it somewhere.

After some work we get $\left| \frac{1}{x} - \frac{1}{2} \right| = \frac{1}{|2x|} |x-2|$.

Would it make sense to begin the proof in the following way?

$$\text{Fix } \varepsilon > 0. \text{ Let } \delta = |2x|\varepsilon \ldots$$

Another tricky example

No, this is nonsense; the x in that expression is undefined/unquantified.

δ must be a single, positive number that can depend on ε, and nothing else.

Back to this expression: $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{1}{|2x|}\,|x-2|$.

We need to get control over the $\frac{1}{|2x|}$ term somehow. If x is allowed to vary freely, $\frac{1}{|2x|}$ can get really big (when x gets close to zero). This seems bad.

But, you have the power to make x as close as you want to 2.

So put a limit on how far x can get from 2, and use that to put an upper bound on $\frac{1}{|2x|}$.

The definition of a limit existing

Here's the precise definition of a limit given in the videos, again.

Definition

Let f be a function defined on an open interval containing a real number c, except possibly at c. Let L be a real number. Then

$$\lim_{x \to c} f(x) = L$$

means

$$\forall \varepsilon > 0 \, \exists \delta > 0 \text{ such that } 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Recall that $\lim_{x \to c} f(x)$ is said to exist if it equals some number L.

Problem. Write a formal definition for $\lim_{x \to c} f(x)$ exists. (This shouldn't take you much time at all.)

Proving limits don't exist

In video 2.8 you saw what it means for a limit to not exist.

Definition

Let $c \in \mathbb{R}$, and let f be a function defined at least on an open interval containing c, except possibly at c.

We say

$$\lim_{x \to c} f(x) \text{ does not exist}$$

if

$$\forall L \in \mathbb{R}, \exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0, \exists x \in \mathbb{R} \text{ s.t. } 0 < |x - c| < \delta \text{ and } |f(x) - L| \geq \varepsilon.$$

You can obtain this by negating the definition we came up with for "$\lim_{x \to c} f(x)$ exists" earlier.

Problem. Prove that $\displaystyle\lim_{x \to 1} \frac{1}{(x-1)^2}$ does not exist.

The limit laws

Theorem (Two basic limits)

For any real numbers c and a:

$$\lim_{x \to c} x = c \quad \text{and} \quad \lim_{x \to c} a = a.$$

Theorem (Limit Laws)

Suppose that

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M.$$

Then

- $\lim_{x \to c} [f(x) + g(x)] = L + M.$
- $\lim_{x \to c} [f(x)g(x)] = LM.$
- $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$, *provided that* $M \neq 0$.

The limits laws

Using the limit laws is very straightforward and natural.

Just make sure that all the initial limits exist. If any of them don't exist, the laws don't apply.

For example, we know that $\lim_{x \to 0} x = 0$. All of the following limits don't exist:

$$\lim_{x \to 0} \frac{1}{x} \qquad \lim_{x \to 0} \frac{1}{\sqrt[3]{x}} \qquad \lim_{x \to 0} \frac{1}{x^3}.$$

But:

$$\lim_{x \to 0} \left[x \cdot \frac{1}{x} \right] = 1 \qquad \lim_{x \to 0} \left[x \cdot \frac{1}{\sqrt[3]{x}} \right] = \infty \qquad \lim_{x \to 0} \left[x \cdot \frac{1}{x^3} \right] = 0.$$