- Course website: `http://uoft.me/MAT137`
- **Remember**: Tutorials start next week. Enrol in one of them!
- Join Piazza, our online help forum. Seriously, it's great.
- Reminder that Problem Set 1 is available, and due 28 September.
  - Next week you'll get an email invitation to submit it online. The instructions for how to do this will be clear.
- For next class: Start watching the first few videos on Playlist 2. I'll update you with exactly how many you need to watch over the weekend.

Domination

(This is a problem from a previous year's first problem set.)

If A and B are both sets of real numbers, we say B *dominates* A if the following is true:

$$\forall a \in A, \exists b \in B, \text{ such that } a < b.$$

Problem. Find two non-empty sets of real numbers A and B such that the following three things are true:

1. $A \cap B = \emptyset$.
2. A dominates B.
3. B dominates A.

One possible solution:

$$A = \{ \text{ even numbers } \} \qquad B = \{ \text{ odd numbers } \}$$

Sets defined with quantifiers

(We didn't talk about this one in lecture, but it's a good exercise to think about.)

Problem. Describe the following sets in the simplest terms you can.

1. $A = \{ x \in \mathbb{R} : \forall y \in [5,7], x < y \}$.
2. $B = \{ x \in \mathbb{R} : \exists y \in [5,7] \text{ such that } x < y \}$
3. $C = \{ x \in [5,7] : \forall y \in [5,7], x < y \}$.

   Think about these ones as homework:

4. $D = \{ x \in [5,7] : \exists y \in [5,7] \text{ such that } x < y \}$
5. $E = \{ x \in [5,7] : \exists y \in \mathbb{R} \text{ such that } x < y \}$

Order of quantifiers matters! A lot!

The following two statements are the same except for the order of the two quantifiers:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } x < y.$$

$$\exists y \in \mathbb{R} \text{ such that } \forall x \in \mathbb{R}, x < y$$

Try to phrase each of these statements as simple English sentences.
Notice that their meanings are **very** different!

A more complicated negation example

Negate the following statement without using any negative words ("no", "not", "none", etc.):

> *"Every page in this book contains at least one word whose first and last letters both come alphabetically before M."*

Negating conditional statements

Here's a great way of thinking about negating conditional statements. Suppose I made you the following promise:

If you get an A (or better) in MAT137, I will give you cake.

Under what circumstances would I have lied to you? Under what circumstances would I have kept my word? For example, if...

- ...you get a C, and I don't give you cake?
- ...I just give everyone cake?
- ...you get an A+, and I don't give you cake?

So what's the negation of the statement?

Conditionals and universal quantifiers

Remember we talked about giraffes...

- There is a purple giraffe in this room. $\longleftarrow$ False
- All giraffes in this room are purple. $\longleftarrow$ True

We used these as examples of how quantification over the empty set works.

The second one can be rewritten as a conditional statement, like this:

- If a giraffe is in this room, it is purple.

This conditional is true because the "if part" is false.

Many conditional statements can be written as "for all" statements, and vice versa.

Problem: Write the following statement as a conditional:

$$\forall x \in \mathbb{Z},\ 2x \text{ is even.}$$

Evens and odds

Write down formal definitions for what it means for an integer to be even or odd.

Which of the following is a correct definition for "odd"? If you think one of them is not a definition of "odd", give a reason why.

1. x is odd if $x = 2n + 1$.
2. x is odd if $\forall n \in \mathbb{Z}$, $x = 2n + 1$.
3. x is odd if $\exists n \in \mathbb{Z}$ such that $x = 2n + 1$.
4. x is odd if $\exists n \in \mathbb{Z}$ such that $x = 2n + 1$.

Having established the definition of oddness, evenness is easy and similar:

$$x \text{ is even if } \exists n \in \mathbb{Z},\ x = 2n.$$

Evens and odds (continued)

Consider the following theorem, which I hope you believe:

Theorem

The sum of two odd integers is even.

Take a moment to think about how you would prove this.

What are some things wrong with the following "proof"?

Proof.

$x = 2a + 1$
$y = 2b + 1$
$x + y = 2n$
$(2a + 1) + (2b + 1) = 2n$
$2(a + b + 1) = 2n$
$a + b + 1 = n.$ □

Evens and odds (continued)

Theorem

The sum of two odd integers is even.

What about the following "proof":

Proof.

For all n:

EVEN + EVEN = EVEN

EVEN + ODD = ODD

ODD + ODD = EVEN

□

I have actually seen someone write this. =(

Write a proof for this statement that is less awful.

Evens and odds (continued)

Theorem

The sum of two odd integers is even.

Here's how I might write a proof of this fact:

Proof.

Let x and y be two odd integers. By the definition of oddness, there must exist two integers n and m such that

$$x = 2n + 1 \quad \text{and} \quad y = 2m + 1.$$

Then we can compute:

$$x + y = (2n + 1) + (2m + 1) = 2n + 2m + 2 = 2(n + m + 1).$$

We know $2(n + m + 1)$ is even by the defininition of evenness, and therefore $x + y$ is even. □

Definitions - Injectivity

A function f defined on a domain D is called injective on D (or sometimes one-to-one on D) if different inputs to the function always yield different outputs.

For example, the function $f(x) = x$ is injective, while the function $g(x) = x^2$ is not.

Try to write down a formal definition for this property.

Definitions - Injectivity (continued)

Here are some candidates.

For all of these, suppose f is a function defined on a nonempty domain D.

1. $f(x_1) \neq f(x_2)$. $\leftarrow$ not a definition.
2. $\forall x_1, x_2 \in D,\ x_1 \neq x_2,\ f(x_1) \neq f(x_2)$. $\leftarrow$ wrong (and bad) definition
3. $\exists x_1, x_2 \in D$ such that $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. $\leftarrow$ every function satisfies this.
4. $\forall x_1, x_2 \in D,\ f(x_1) \neq f(x_2) \implies x_1 \neq x_2$. $\leftarrow$ definition of "f is a function".
5. $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$.
6. $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.
7. $\forall x_1, x_2 \in D,\ f(x_1) = f(x_2) \implies x_1 = x_2$.
8. $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

A simple proof

In one of the videos, you learned the definition of an increasing function. Here it is again, for reference:

Definition

A function f is increasing on an interval D if

$$\forall x_1, x_2 \in D, \; x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$$

Prove the following theorem:

Theorem

If a function f is increasing on D, then it is injective on D.