MAT137 - Term 2, Week 12

- Last class!
- Your last problem set is due **Wednesday**, April 5 at 3pm.
- Course evaluations are now available.
  Please fill one out. It's important to me.
- Today we will:
  - Remind ourselves a bit about Taylor series
  - Use Taylor series to do cool things.

A word I didn't define last time

There's a word I forgot to define last time, that you may have seen on PS10:

Definition

The Taylor series of a function f centred at $a = 0$ is called its <u>Maclaurin series</u>.

This is just a name, and doesn't add any meaning. I always just say "Taylor series centred at 0".

Taylor series we know so far

Here are the six most important Taylor series we know so far.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad (R = \infty)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (R = 1)$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad (R = \infty)$$

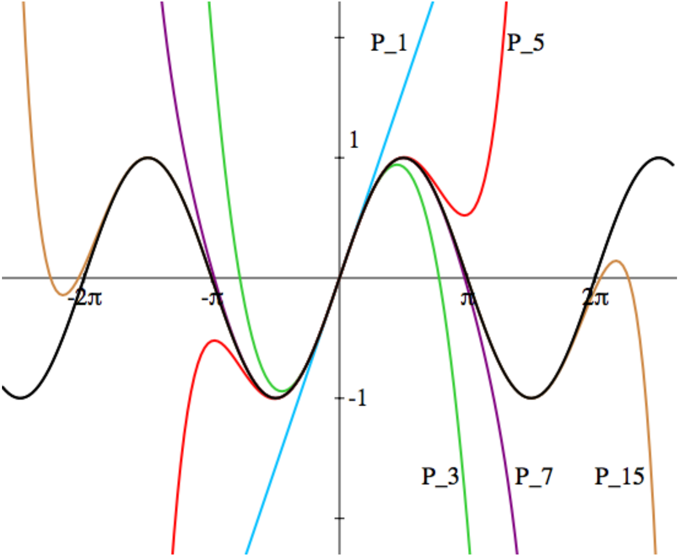
$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad (R = 1)$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad (R = \infty)$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \qquad (R = 1)$$

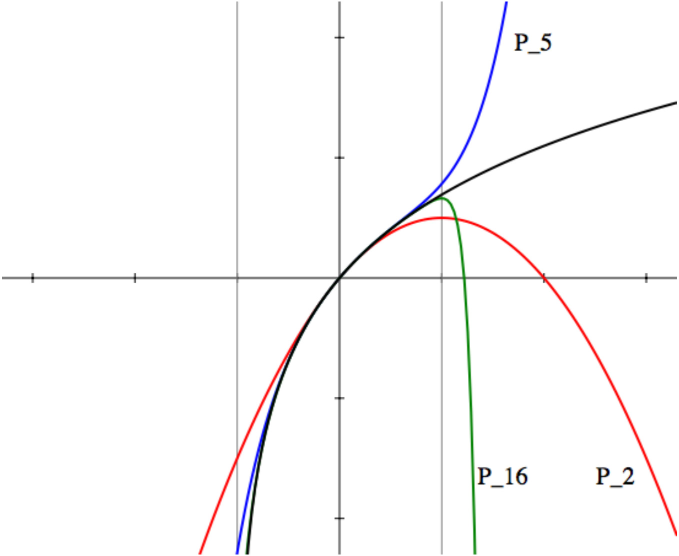
Some pictures

Here is a graph of $\sin(x)$, and some of its Taylor polynomials.



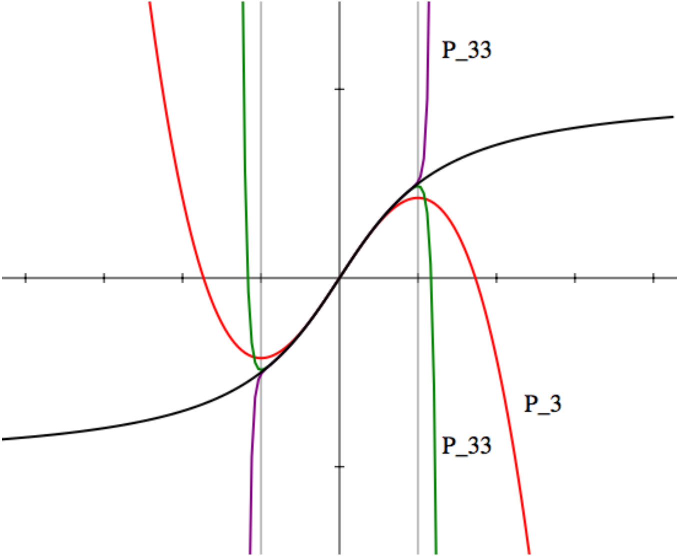
Some pictures

Here is a graph of $\log(1+x)$, and some of its Taylor polynomials.



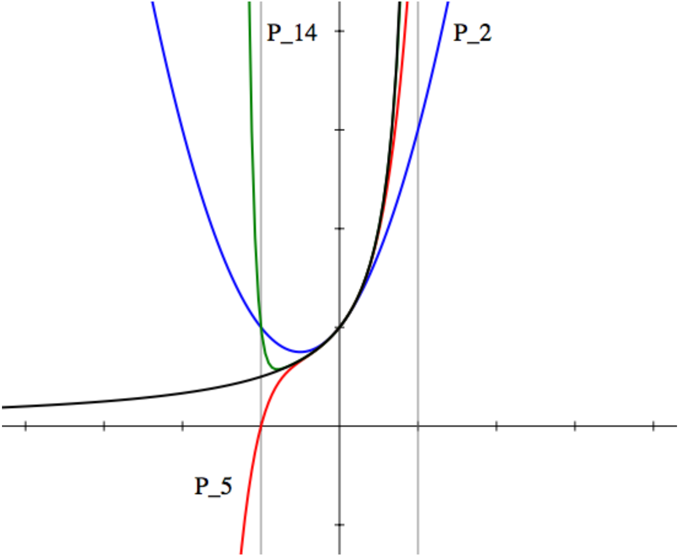
Some pictures

Here is a graph of $\arctan(x)$, and some of its Taylor polynomials.



Some pictures

Here is a graph of $\frac{1}{1-x}$, and some of its Taylor polynomials.



Some pictures

Notice how you can see in the last three graphs that the function is only approximated by its Taylor polynomials between -1 and 1.

As we know, the Taylor *series* of these functions don't even converge outside $[-1, 1]$.

Using Taylor series to evaluate limits

Taylor series are very, very useful for evaluating limits.

This is the thing your instructors and TAs usually do in their head when given a tricky limit.

Example: Compute the following limit:

$$\lim_{x \to 0} \frac{\cos(x) - 1 + \frac{1}{2}x^2 e^x}{x^3}.$$

This can be done with L'Hôpital's rule, but it's much easier with Taylor series.

Using Taylor series to evaluate limits

$$\lim_{x \to 0} \frac{\cos(x) - 1 + \frac{1}{2} x^2 e^x}{x^3}.$$

Taylor series of the numerator:

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right) - 1 + \frac{1}{2} x^2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\right)$$

After distributing the coefficient in front of the last pair of brackets, this looks like:

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right) - 1 + \left(\frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{2 \cdot 2} + \frac{x^5}{2 \cdot 3!} + \cdots\right)$$

Using Taylor series to evaluate limits

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right) - 1 + \left(\frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{2\cdot 2} + \frac{x^5}{2\cdot 3!} + \cdots\right)$$

We notice that the constant and quadratic terms cancel out to leave:

$$\frac{x^3}{2} + \left(\frac{1}{24} + \frac{1}{4}\right)x^4 + \text{higher degree terms (which won't matter)}$$

As a matter of convention, we "gather up" higher degree terms that don't matter like this:

$$\frac{x^3}{2} + \frac{7}{24}x^4 + (\text{CONSTANT})x^5 + \cdots = \frac{x^3}{2} + \frac{7}{24}x^4 + O(x^5)$$

Using Taylor series to evaluate limits

So we've determined that the Taylor series of the numerator starts off like this:

$$\frac{x^3}{2} + \frac{7}{24}x^4 + O(x^5)$$

Of course, what we're actually trying to do is calculate the limit of this over x^3 at zero.

So the Taylor series of the whole quotient is:

$$\frac{\cos(x) - 1 + \frac{1}{2}x^2e^x}{x^3} = \frac{\frac{x^3}{2} + \frac{7}{24}x^4 + O(x^5)}{x^3} = \frac{1}{2} + \frac{7}{24}x + O(x^2)$$

Once we know this, it's easy to simply read off the limit:

$$\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{1}{2}x^2e^x}{x^3} = \lim_{x\to 0} \frac{1}{2} + \frac{7}{24}x + O(x^2) = \frac{1}{2}$$

Another example

Example: Compute the following limit:

$$\lim_{x \to 0} \frac{\cos(x) - 1 + \frac{1}{2} x \sin(x)}{(\log(1+x))^4}.$$

This is a tiny bit trickier. Before we could easily tell that we only needed the x^3 term in the numerator, because the denominator was x^3.

In this case we can first determine the lowest-degree term in the denominator.

$$(\log(1+x))^4 = \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \right)^4 = x^4 + O(x^5)$$

So all we need in the numerator are the terms up to degree 4.

Another example

Example: Compute the following limit:

$$\lim_{x\to 0} \frac{\left(x\, e^{x} + \sin(3x) - x^2\right)(\cos(x) - 1)}{2\sin(x^2)(e^{\pi\, x} - 1)}$$

Examples

What function is this the Taylor series of?

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+2}$$

We notice that $\frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+2}$ is an antiderivative of

$$\frac{(-1)^n}{(2n+1)} x^{2n+1}.$$

These are the terms in the Taylor series of $\arctan(x)$. So our original series is the Taylor series of

$$\int \arctan(x)\, dx = x \arctan(x) - \frac{1}{2} \log(x^2+1).$$

(The constant of integration here is 0.)

Example

Now what about the following series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+7}$$

Well if we factor out x^5, we get the previous series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+7} = x^5 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+2}$$

So this is the Taylor series of the function:

$$x^5 \int \arctan(x)\, dx = x^5 \left(x \arctan(x) - \frac{1}{2} \log(x^2+1) \right)$$

Example

These are the sorts of manipulation we use to compute the sums of series using Taylor series.

Example: Compute the value of the sum: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)}$.

We can see that this series can be obtained by substituting $x = 1$ into the Taylor series from earlier:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} (1)^{2n+2}$$

Since we know what function the Taylor series represents, we can compute:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} (1)^{2n+2} = (1) \arctan(1) - \frac{1}{2} \log\left((1)^2 + 1\right) = \frac{\pi}{4} - \frac{\log(2)}{2}$$

Examples of summing series

Example: Compute: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^{n+1}}$.

Example: Compute $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\pi^{2n}}{4^n\,(2n+1)!}$.

Integrating with Taylor series

You've probably heard that the function e^{-x^2} has no "elementary" antiderivative.

With Taylor series, we can approximate this function with a polynomial to whatever precision we want though:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}.$$

So if you want to know the value of $\int_0^{10} e^{-x^2}\, dx$, you can approximate it as well as you want, since:

$$\int_0^x e^{-t^2}\, dt = \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} \right) dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\, n!} x^{2n+1}.$$

So the answer to your question is the value of this series at $x = 10$, and you can get a bound on the error because it's an alternating series.

Integrating with Taylor series

By the way, here's an astonishing fact:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Integrating with Taylor series

Example: Compute $\int \frac{x^5 + 7x^3 + 2x^2 + x + 1}{(x-1)^4}\,dx$.

I like this example because it makes use of the trivial-seeming fact that polynomials are analytic.

What about tangent?

The Taylor series for tangent is very difficult to write down in a nice, closed form.

Example: Write down the first few terms of the Taylor series for $\tan(x)$ at 0.

From the definition of tangent, we know that $\sin(x) = \tan(x)\cos(x)$. If we assume $\tan(x)$ can be written as a Taylor series $\sum_{n=0}^{\infty} a_n x^n$, then this equation can be written:

$$\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) = (a_0 + a_1 x + \cdots)\left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots\right).$$

What about tangent?

$$\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) = (a_0 + a_1 x + \cdots)\left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots\right).$$

The two power series on the right can then be multiplied together to form a single power series. Doing this for every term is complicated, but the first few are easy. You get:

$$a_0 + a_1 x + \left(a_2 - \frac{a_0}{2}\right)x^2 + \left(a_3 + \frac{a_1}{2}\right)x^3 + \left(a_4 + \frac{a_0}{4!} + \frac{a_2}{2}\right)x^4 + \cdots$$

Equating this to the Taylor series for cosine allows us to solve for some of the a_i's:

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = \frac{1}{3}, \quad a_4 = 0, \ldots$$