MAT137 - Term 2, Week 9

- Problem Set 9 is due next Friday, 17 March, at 3pm.
  Don't leave it for the last few days.

- Today we will:
  - Remind ourselves about definitions and results from last class.
  - Talk about the Ratio Test for series.
  - Talk about absolute convergence, conditional convergence, and alternating series.
  - Introduce the topic of power series, if there's time.

Definition from last class

Definition

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. For each n, define the <u>n^{th} partial sum</u> of the sequence by:

$$s_n = \sum_{k=0}^{n} a_k.$$

That is, $s_n = a_0 + a_1 + a_2 + \cdots + a_n$.

If $\lim_{n\to\infty} s_n$ exists and equals a number L, we say that the series $\sum_{n=0}^{\infty} a_n$ <u>converges to L</u>, and call L the <u>sum</u> of the series.

If the limit of the sequence above does not exist, we say that $\sum_{n=0}^{\infty} a_n$ <u>diverges</u>.

Results from last class

Theorem (Necessary Condition Test (NCT))

Suppose $\{a_n\}_{n=0}^{\infty}$ *is a sequence. If* $\sum_{n=0}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Or its contrapositive form:

$$\text{If} \quad \lim_{n\to\infty} a_n \neq 0, \quad \text{then} \quad \sum_{n=0}^{\infty} a_n \quad \text{diverges.}$$

This allowed us to easily determine that many series diverge.

Results from last class

Remember that the *converse* of the NCT is not true:

$$\text{If } \lim_{n\to\infty} a_n = 0, \text{ it does not follow that } \sum_{n=0}^{\infty} a_n \text{ converges.}$$

Proof: The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, even though $\frac{1}{n} \to 0$.

Results from last class

This was exactly analogous to a result we had for improper integrals:

Proposition

If $\{a_n\}_{n=0}^{\infty}$ is a sequence of positive numbers, then the

$$\sum_{n=0}^{\infty} a_n$$

either converges, or diverges to infinity.

Results from last class

We talked about two specific types of series:

Telescoping series, like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$.

For these, we can explicitly simplify s_n and take a limit easily because of cancellation.

We also talked about geometric series, which are very important:

Theorem

The geometric series $\displaystyle\sum_{n=0}^{\infty} r^n$ *converges if and only if* $|r| < 1$.

In this case,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Last class: The Integral Test

Theorem

Suppose f is a continuous, positive, decreasing function defined on $[1, \infty)$.

Then

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges.}$$

NOTE: This doesn't say the series *equals* the integral.

A consequence is the p-**test**: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Last class: The Basic Comparison Test

Theorem (Basic Comparison Test (BCT))

Suppose that $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are sequences with* <u>*positive*</u> *terms.*

Suppose also that $a_n \leq b_n$ *for all* n.

1. *If* $\sum_{n=0}^{\infty} b_n$ *converges, then* $\sum_{n=0}^{\infty} a_n$ *converges.*
2. *If* $\sum_{n=0}^{\infty} a_n$ *diverges, then* $\sum_{n=0}^{\infty} b_n$ *diverges.*

Last class: The Limit Comparison Test

Theorem (Limit Comparison Test (LCT))

Suppose that $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are sequences with* <u>*positive*</u> *terms.*

Suppose also that $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists and equals a positive constant.*

Then

$$\sum_{n=0}^{\infty} a_n \text{ converges} \quad \iff \quad \sum_{n=0}^{\infty} b_n \text{ converges.}$$

Another way to state the conclusion of this theorem is that the two series either both converge or both diverge.

The Ratio Test

This is the test you will end up using the most towards the end of this course.

Idea: Design a test that (informally) compares a given series $\sum a_n$ to a geometric series, and then uses what we know about geometric series to conclude something about the original series.

Observe that for a geometric series $\sum r^n$, we always have that the ratio of two consecutive terms is constant:

$$\frac{r^{n+1}}{r^n} = r.$$

For a general series we can't expect this, of course, but we can ask about the limit of this ratio.

The Ratio Test

Theorem (Ratio Test)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with positive terms.

Suppose also that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists, and equals a real number λ.

Then:

1. *If $\lambda < 1$, then the series $\sum a_n$ converges.*
2. *If $\lambda > 1$, then the series $\sum a_n$ diverges.*
3. *If $\lambda = 1$, we can't conclude anything about $\sum a_n$.*

Note: Note that $\lambda = 0$ is fine here, in contrast to the LCT. Don't mix them up!

Examples

Example: $\sum \frac{1}{n!}$. This converges by the ratio test.

Example: $\sum \frac{(n-1)!}{2^n (n+1)^2}$. This diverges by the ratio test (but it isn't hard to see without it).

Example: $\sum \frac{\log(n)}{n}$. This diverges by the integral test or BCT, but the ratio test is inconclusive.

Example: $\sum \frac{\pi\, n^7 \log(n^2)}{7^n}$. This converges by the ratio test.

Absolute Convergence

So far most of our results have been about series with positive terms. This is not by accident.

Definition

A series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ converges.

This notion is strictly stronger than regular convergence, as the next two results will illustrate. First of all:

Theorem

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

ie. Absolute convergent series are convergent.

Absolute convergence

Note that any convergent series with positive terms is absolutely convergent. This is a deceptively powerful fact.

Example: $\sum \frac{(-1)^n}{n}$ is convergent (a fact we can't prove easily yet, but will prove soon).

On the other hand, $\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n}$, which we know diverges.

The series above is called the alternating harmonic series.

Definition

A series which is convergent but not absolutely convergent is called conditionally convergent.

Alternating series

The most interesting series that involve negative terms are the so-called alternating series. These are series in which every other term is negative.

Definition

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence with positive terms.

Then the series $\sum (-1)^n a_n$ is called an alternating series.

These series will come up very often when we talk about Taylor series.

Alternating Series test

The great thing about alternating series is that there's a very simple convergence test for them.

Theorem (Alternating series test)

Let $\{a_n\}_{n=0}^{\infty}$ be a decreasing sequence with positive terms.

Suppose also that $\lim_{n\to\infty} a_n = 0$.

Then $\sum (-1)^n a_n$ converges.

The proof for this is a straightforward, but tedious, application of the Monotone Sequence Theorem. You are encouraged to read it in your textbook.

Examples

Example: As promised earlier, $\sum \frac{(-1)^n}{n}$ converges by the AST. Therefore, this series is conditionally convergent.

Example: $\sum \frac{(-1)^n}{n \log(n)}$ converges by the AST.

It is also conditionally convergent, since $\sum \frac{1}{n \log(n)}$ diverges by the integral test.

A useful thing to note (for the future):

$$(-1)^n = \cos(n\pi) = \sin(n\pi - \tfrac{\pi}{2}).$$

Estimations based on alternating series

Another very useful feature of alternating series is that they allow us to estimate things easily.

Note that if $\{a_n\}$ is a decreasing sequence of positive numbers, then:

- The sequence $\{s_{2n}\}$ of partial sums with even index is decreasing.
- The sequence $\{s_{2n+1}\}$ of partial sums with odd index is increasing.

Because of this, if $\lim_{n\to\infty} s_n$ exists and equals L, we must have that

$$s_n < L < s_{n+1} \text{ for all odd } n, \text{ and } s_{n+1} < L < s_n \text{ for all even } n$$

As a result, we can conclude:

$$|s_n - L| < |s_{n+1} - s_n| = a_{n+1}$$

Estimating

$$|s_n - L| < |s_{n+1} - s_n| = a_{n+1}$$

What does this all mean?

The sum of the first n terms of a convergent alternating series with decreasing terms differs from from the sum of the whole series by at most a_{n+1}.

Example

Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

This series converges by the AST, and the positive part of its terms clearly decrease. Call its sum L We'd like to give an estimate of L, correct to within 10^{-6}.

By the result from earlier, we know:

$$\left| \sum_{k=0}^{n} \frac{(-1)^k}{k!} - L \right| < \frac{1}{(n+1)!}.$$

In other words, we want the smallest n such that $(n+1)! > 1\,000\,000$.

We can compute $9! = 362\,880$, so $10! > 1\,000\,000$. Therefore:

$$\left| \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{9!} \right) - L \right| < 10^{-6}.$$