MAT137 - Term 2, Week 7

- Problem Set 7 is due tomorrow, 17 February, at 3pm.
- Today we will:
  - Talk a bit more about the Monotone Sequence Theorem, and see how it can be used.
  - Talk about "The Big Theorem".
  - Talk about improper integrals.

Two main theorems

To remind you, we have two main theorems about sequences.

These results describe how boundedness, monotonicity, and convergence relate to one another.

The first one is pretty easy to prove, and you basically proved the second one on Problem Set 6 without realizing it.

Theorem

Every convergent sequence is bounded.

The contrapositive of this theorem is useful enough to be worth stating:

Remark

Every unbounded sequence diverges.

Monotone sequence theorem

Theorem (Monotone Sequence Theorem)

If $\{a_n\}_{n=1}^{\infty}$ is a bounded, monotonic sequence, then it converges.

More specifically, if the sequence is non-decreasing, then it converges to $\sup\{\, a_n \;:\; n \in \mathbb{N} \,\}$.

Similarly, if the sequence is non-increasing, then it converges to $\inf\{\, a_n \;:\; n \in \mathbb{N} \,\}$.

Using the MST

Note that the MST doesn't really help you compute limits. It just tells you limits exist sometimes.

It *does* tell you what value the limit should equal (a certain supremum or infimum), but computing that value usually involves doing mostly the same work as proving the limit does.

The MST is a very important theorem, but its value is largely theoretical. On the next slide we'll see an example of its use.

Using the MST

Recall that we proved earlier that the sequence defined recursively by

$$b_1 = 1 \quad \text{and} \quad b_{n+1} = \sqrt[3]{b_n + 6}.$$

is bounded and monotonic.

By the MST, we now know it converges. We can use this knowledge to compute the limit explicitly.

Suppose it converges to L. Then we have:

$$b_{n+1} = \sqrt[3]{b_n + 6} \quad \Longrightarrow \quad \lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} \sqrt[3]{b_n + 6}.$$

Since the cube root function is continuous, this gives us:

$$L = \sqrt[3]{L + 6} \quad \Longrightarrow \quad L^3 = L + 6 \quad \Longrightarrow \quad L = 2.$$

You really need the MST there

Without knowing the sequence $\{b_n\}$ converges (by the MST), the proof above doesn't work. Here's an example to illustrate that.

Recursively define a sequence by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = 1 - a_n.$$

Suppose the sequence converges to L. Using the same sort of technique as before, we get

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 1 - a_n. \quad \implies \quad L = 1 - L \quad \implies \quad L = \frac{1}{2}.$$

This all seems fine, but if you pay attention to the actual sequence, its values are

$$1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

which obviously doesn't converge.

The Big Theorem

Of all the things we learn about sequences, this will probably be the most useful.

This theorem allows us to compute many limits very easily.

For example, recall this sort of limit from the first term:

$$\lim_{x\to\infty} \frac{7e^x + 12x^4 + \pi \log(x)}{10x^7 + 2e^x}.$$

This looks complex, but we learned that the exponential function $f(x) = e^x$ grows much faster than any polynomial or any logarithm. That is, we were able to prove (with l'Hopital's rule) that:

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0 \text{ for any positive } n, \text{ and} \qquad \lim_{x\to\infty} \frac{\log(x)}{e^x} = 0.$$

The Big Theorem

Using these two facts, computing the scary-looking limit from before is easy:

$$\lim_{x\to\infty} \frac{7e^x + 12x^4 + \pi \log(x)}{10x^7 + 2e^x} = \lim_{x\to\infty} \frac{e^x \left(7 + 12\frac{x^4}{e^x} + \pi \frac{\log(x)}{e^x}\right)}{e^x \left(10\frac{x^7}{e^x} + 2\right)} = \frac{7}{2}.$$

The sort of calculation we did above works just as well for sequences involving logarithms, exponentials, and polynomials. So for example, the same proof will show that:

$$\lim_{n\to\infty} \frac{7e^n + 12n^4 + \pi \log(n)}{10n^7 + 2e^n} = \frac{7}{2}.$$

The "Big Theorem" generalizes this result a bit, to take into account sequences like $n!$ and n^n that either aren't possible with functions, or rarely come up with functions.

Some notation before The Big Theorem

Some notation that will make these things easier to talk about.

Definition

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of *positive* numbers.

We say that $\underline{a_n \text{ is much smaller than } b_n}$, or $\underline{b_n \text{ grows much faster than } a_n}$, if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

If b_n grows much faster than a_n, we denote it by writing

$$a_n << b_n.$$

Computer scientists may be familiar with "little-*o* notation":

$$a_n << b_n \quad \iff \quad a_n \in o(b_n).$$

Preliminaries

Here's a rather obvious fact that we should get out of the way:

Proposition

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be sequences of positive numbers.

If $a_n << b_n$ and $b_n << c_n$, then $a_n << c_n$.

The proof of this is left as an easy exercise for the reader.

A mathematician would express this result by saying "the relation $<<$ is transitive".

The Big Theorem

Theorem

For any positive number a, and any real number $c > 1$,

$$\log(n) << n^a << c^n << n! << n^n.$$

Note: I've used a natural logarithm in the statement above, but any base larger than 1 will work since for any $b > 1$

$$\log_b(n) = \frac{\log(n)}{\log(b)}.$$

Proving this theorem amounts to proving four $<<$ relationships.

The first two we already know to be true about functions, and the same proofs work here.

Exercise: Prove the first two $<<$ relationships from the theorem.

The Big Theorem

Proofs for the other two $<<$ relationships can be found in a video we posted to the course website this afternoon.

I'll just give you the ideas behind these results.

Proposition

For any $c > 1$, $c^n << n!$. In other words, $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

Idea: Fix any $c > 1$, and let $a_n = \frac{c^n}{n!}$. So we want to show that $\lim_{n\to\infty} a_n = 0$.
Note that

$$a_{n+1} = \frac{c}{n+1} a_n.$$

Use this relation to show that a_n is eventually decreasing and bounded below, then use the MST to compute the limit.

The Big Theorem

Proposition

$n! << n^n$. In other words, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Idea: Note that:

$$\frac{n!}{n^n} \quad = \quad \frac{1}{n}\,\frac{2}{n}\,\frac{3}{n}\cdots\frac{n-1}{n}\,\frac{n}{n} \quad = \quad \frac{1}{n}\left(\frac{2}{n}\,\frac{3}{n}\cdots\frac{n-1}{n}\,\frac{n}{n}\right) \quad \leq \quad \frac{1}{n}.$$

Now use the Squeeze Theorem.

Using the Big Theorem

Having proved this result, many horrific-looking limits are now trivial to compute.

Example: Compute $\lim\limits_{n\to\infty} \dfrac{7n^{12}\,\log_{88}(n^2)\,n!}{5(n+1)^{\pi}\,(3n)^n}$.

Simply rearrange things to see that:

$$\lim_{n\to\infty} \frac{7n^{12}\,\log_{88}(n^2)\,n!}{5(n+1)^{\pi}\,(3n)^n} = \lim_{n\to\infty} \frac{7}{5}\cdot\frac{n^{12}}{3^n}\cdot\frac{2\log_{88}(n)}{(n+1)^{\pi}}\cdot\frac{n!}{n^n}.$$

Each of the three fractions involving n on the right limit to zero by the Big Theorem, so the whole limit is zero.

Improper integrals

Every definite integral we've discussed so far in this course has computed the (signed) area underneath the graph of a function f, on an interval of the form $[a, b]$ on which f is defined and bounded.

The goal of this section is to extend our ideas about bounded functions on closed intervals to open intervals, or unbounded functions.

For example, our methodology from before can't help us compute something like

$$\int_1^\infty f(x)\,dx,$$

since an approximation by rectangles would involve infinitely many rectangles, and we can't add up infinitely many numbers.

Improper integrals

Our solution will be to approximate the area we want with things we do know.

For example, suppose we want to compute the area under the graph of $\frac{1}{x^2}$ on the interval $[1, \infty)$.

We will "approximate" it by the area under the graph on an interval $[1, b]$ for some $b > 1$, then see what happens as we increase b.

Improper integrals

Definition

Let f be a function defined on an interval $[a, \infty)$ for some $a \in \mathbb{R}$, and such that f is integrable on $[a, b]$ for any $b > a$.

Then we define an improper integral by:

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \left[\int_a^b f(x)\,dx \right].$$

If this limit exists, we say that the improper integral converges, and that it diverges otherwise.

We can make a similar definition and say:

$$\int_{-\infty}^a f(x)\,dx = \lim_{b \to -\infty} \left[\int_b^a f(x)\,dx \right].$$

Example

Example: Determine whether $\int_1^\infty \frac{1}{x^2}\,dx$ converges, and compute its value if it does.

By definition, we have

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{1}{x^2}\,dx\right] = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} -\frac{1}{b}+1 = 1.$$

An important example

This exercise generalizes the result of the previous exercise, and gives us a very important result we will use many times for the rest of the year.

Example: For which $p > 1$ does the integral $\int_1^\infty \frac{1}{x^p}\,dx$ converge?
Applying the definition, we get:

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{1}{x^p}\,dx\right] = \lim_{b\to\infty}\left[\frac{1}{1-p}\,\frac{1}{x^{p-1}}\right]_1^b,$$

(which only makes sense if $p \neq 1$).

So our integral converges exactly when the limit:

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} - 1$$

exists.

An important example, continued

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} - 1$$

This limit exists if and only if $p - 1 > 0$, or $p > 1$.

The $p = 1$ case we can examine separately, and easily conclude that $\int_1^\infty \frac{1}{x}\,dx$ diverges.

So we have proved this fact:

Proposition

The improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges for all $p > 1$, and diverges for all $0 < p \leq 1$.

Another example

Example: Determine whether $\int_0^\infty \cos(x)\, dx$ converges.

It doesn't, since $\lim_{b\to\infty} \sin(b)$ does not exist.

This leads us to notice a fact, whose proof is easy to see given what we've done so far.

Proposition

Let f be a positive function defined on $[a, \infty)$ and integrable everywhere necessary.

Then $\int_a^\infty f(x)\, dx$ either converges, or diverges to infinity.

ie. If f is positive, it can't do what cosine did above.

Integrals over all of $\mathbb{R}$

The same idea can be used to define integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$, but we have to be slightly careful.

It seems like we might want to say:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b\to\infty}\left[\int_{-b}^{b} f(x)\,dx\right],$$

but this can get you into trouble.

For example, if we defined it this way we would get $\int_{-\infty}^{\infty} \sin(x)\,dx = 0$, which should seem problematic given our earlier result about the improper integral of $\cos(x)$.

We have to separate out the two halves.

Integrals over all of $\mathbb{R}$

Definition

Let f be integrable on every interval of the form $[a, b]$. Then we say the improper integral

$$\int_{-\infty}^{\infty} f(x)\, dx$$

converges if

$$\int_{-\infty}^{1} f(x)\, dx \quad \text{and} \quad \int_{1}^{\infty} f(x)\, dx$$

both converge.

If they do, $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx$ equals their sum.

Example: $\displaystyle\int_{-\infty}^{\infty} \cos(x)\, dx$ and $\displaystyle\int_{-\infty}^{\infty} \sin(x)\, dx$ do not converge.

Unbounded functions

We just saw how to extend our definition of integrability to unbounded intervals like $[a, \infty)$. What about functions that become unbounded at a point?

What's wrong with the following computation:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -\frac{1}{1} + \frac{1}{-1} = -2.$$

Answer: $\frac{1}{x^2}$ is not integrable on $[-1, 1]$, so the definite integral here just doesn't make sense to write down.

However, $\frac{1}{x^2}$ *is* integrable on $[-1, -\epsilon]$ and $[\epsilon, 1]$ for any $\epsilon > 0$. So again we try to define integrals like this by approximating them.

Integrals of unbounded functions

Definition

Suppose f is defined on an interval $[a, b)$, but becomes unbounded near b. ie.

$$\lim_{x \to b^-} f(x) = \pm\infty.$$

Suppose also that f is integrable on $[a, x]$ for all $a < x < b$. Then we define:

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \left[\int_a^t f(x)\,dx \right].$$

As before, we say that the integral <u>converges</u> if this limit exists, and <u>diverges</u> otherwise.

As you can imagine, we can make an analogous definition when f becomes unbounded at the left endpoint of an interval.

Examples

Example: Compute the following two improper integrals:

$$\int_0^1 \frac{1}{x^2}\,dx \quad \text{and} \quad \int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

Homework: For which values of $p > 0$ does $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converge?

Comparison tests

Determining whether an improper integral converges or diverges can be hard, because finding antiderivatives can be hard.

We'd like to make it easier to determine whether an integral converges or not. In particular, we want to be able to use things we know about simple functions to say things about more complicated functions.

Here, we'll develop two powerful tools for doing this. These tools will be *very* useful for us when we talk about series as well, so pay close attention!

The Basic Comparison Test

Theorem (Basic Comparison Test (BCT))

Let $a \in \mathbb{R}$, and let f, g be functions that are integrable on $[a, b]$ for every $b > a$.

Suppose also that $0 \leq f(x) \leq g(x)$ for all $x \in [a, \infty)$. Then

1. *If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges as well.*
2. *If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges as well.*

Homework: Prove this theorem. Hint for (1):

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx \leq \int_a^\infty g(x)\,dx$$

Now use a result similar to Problem 1 on Problem Set 6.

Example

Example: Determine whether $\displaystyle\int_1^{\infty} \frac{1}{x+e^x}\,dx$ converges or diverges.

Note that we don't know how to intergrate this function, so we can't just apply the definition of improper integrals.

However, two inequalities seem to come to mind here. For $x \geq 1$, we know that

$$\frac{1}{x+e^x} \leq \frac{1}{x} \quad \text{and} \quad \frac{1}{x+e^x} \leq \frac{1}{e^x}.$$

One of these is helpful, and the other is not.