MAT137 - Term 2, Week 6

- Problem Set 7 is due next Friday, 17 February, at 3pm.
- Today we will:
  - Talk a bit more about trigonometric substitutions.
  - Spend the rest of the time talking about sequences.

Trigonometric substitutions

To remind you, our tool here is the Pythagorean Identity, in the following three forms:

- $1 - \sin^2(x) = \cos^2(x)$.
- $1 + \tan^2(x) = \sec^2(x)$.
- $\sec^2(x) - 1 = \tan^2(x)$.

These will allow us to simplify many integrals that include terms of the following three forms:

- $a^2 - c^2x^2$.
- $a^2 + c^2x^2$.
- $c^2x^2 - a^2$

by making substitutions which cause the left sides of the respective identities to appear.

Trigonometric substitutions

For example, if a term like $a^2 + c^2x^2$ appears in an integral, we may want to make the substitution

$$x = \frac{a}{c} \tan\theta,$$

so that this term will become

$$a^2 + c^2 \left(\frac{a}{c} \tan\theta\right)^2 = a^2(1 + \tan^2\theta) = a^2 \sec^2\theta$$

after substituting.

This is the only new idea here. Everything else is just “bookkeeping”.

Trigonometric substitution

Example: Compute $\int \frac{\sqrt{9x^2 - 4}}{x}\, dx$.

Take a moment to determine what the substitution should be.

You should have found the substitution $x = \frac{2}{3}\sec\theta$. With this substitution, we compute:

$$\frac{dx}{d\theta} = \frac{2}{3}\sec\theta\tan\theta \quad \Rightarrow \quad dx = \frac{2}{3}\sec\theta\tan\theta\, d\theta.$$

Trigonometric substitution

Now let's plug all of this into the integral:

$$\int \frac{\sqrt{9x^2 - 4}}{x}\, dx = \int \frac{\sqrt{9\left(\frac{2}{3}\sec\theta\right)^2 - 4}}{\left(\frac{2}{3}\sec\theta\right)} \frac{2}{3}\sec\theta\tan\theta\, d\theta.$$

This looks scary, but many things cancel to give:

$$\int \sqrt{4\sec^2\theta - 4}\,\tan\theta\, d\theta = \int 2\sqrt{\tan^2\theta}\,\tan\theta\, d\theta.$$

This simplification inside the square root due to the identity

$$\sec^2\theta - 1 = \tan^2\theta$$

is the reason we made the substitution.

Trigonometric substitutions

So we're here:

$$\int 2\sqrt{\tan^2\theta}\,\tan\theta\,d\theta = 2\int \tan^2\theta\,d\theta.$$

I'm hiding something about the sign of $\tan\theta$ here. We'll return to this shortly.

From here, we can finish computing the antiderivative using conventional means:

$$= 2\int \sec^2\theta - 1\,d\theta = 2\left[\tan\theta - \theta\right] + C$$

Finally, we put this back in terms of x.

$$2\left[\tan\theta - \theta\right] + C$$

Our substitution was $x = \frac{2}{3}\sec\theta$, so $\theta = \operatorname{arcsec}\left(\frac{3}{2}x\right)$. By looking at a triangle we can compute:

$$\tan\theta = \frac{\sqrt{9x^2 - 4}}{2}.$$

So our final answer is:

$$\int \frac{\sqrt{9x^2 - 4}}{x}\,dx = 2\left[\tan\theta - \theta\right] + C = \sqrt{9x^2 - 4} - 2\operatorname{arcsec}\left(\frac{3}{2}x\right) + C.$$

Trigonometric substitution

Above we assumed that $\tan\theta$ was positive. Why is that okay?

Well, it wasn't. It turns out we didn't give a complete answer.

If we look at the original integrand $\frac{\sqrt{9x^2-4}}{x}$ we see that x must be either larger than $\frac{2}{3}$ or less than $-\frac{2}{3}$.

When x is positive, we can represent it as $x = \frac{2}{3}\sec\theta$ for values of θ in $[0, \frac{\pi}{2})$. For these θ's, $\tan\theta$ is positive, so there's no problem.

Trigonometric substitutions

What about when x is negative? In this case, we have to change things slightly.

In this case, when we say $x = \frac{2}{3}\sec\theta$, we want values of θ in $(\frac{\pi}{2}, \pi]$. For these θ's, $\tan\theta$ is negative.

So when we get to this stage in our computation:

$$2\int \sqrt{\tan^2\theta}\tan\theta\, d\theta,$$

we note that this equals

$$-2\int \tan^2\theta\, d\theta.$$

This leads to the following final answer:

$$\int \frac{\sqrt{9x^2-4}}{x}\,dx = \sqrt{9x^2-4} + 2\,\mathrm{arcsec}\left(\frac{3}{2}x\right) + C.$$

Trigonometric substitutions

Another example, to illustrate how any quadratic function in an integral can be attacked with trig substitutions.

Example: Compute $\int \frac{x}{\sqrt{x^2+2x+2}}\,dx$.

Here we don't see anything that looks like one of the Pythagorean identities. But it's there, hiding.

First, complete the square in the quadratic in the denominator:

$$x^2+2x+2=(x+1)^2+1.$$

Trigonometric substitutions

Now our integral looks like:

$$\int \frac{x}{\sqrt{(x+1)^2+1.}}\,dx$$

The quadratic in the denominator *does* look like the identity $\tan^2 x + 1 = \sec^2 x$ now!

So we make the substitution $x + 1 = \tan\theta$.

Then

$$dx = \sec^2\theta\,d\theta \quad \text{and} \quad x = \tan\theta - 1.$$

Trigonometric substitutions

Substituting these things into our original integral, we get:

$$\int \frac{\tan\theta - 1}{\sqrt{\tan^2\theta + 1}} \sec^2\theta \, d\theta = \int \frac{\tan\theta - 1}{\sqrt{\sec^2\theta}} \sec^2\theta \, d\theta.$$

From here we can proceed as normal:

$$= \int \frac{\tan\theta - 1}{\sec\theta} \sec^2\theta \, d\theta = \int \tan\theta \sec\theta - \sec\theta \, d\theta.$$

And we've successfully reduced the problem to an integral we can do.

Sequences

A *sequence of real numbers* is an infinite list of real numbers written in a specific order, like this:

$$a_1, a_2, a_3, a_4, \ldots$$

We call a_1 the "first term" of the sequence, a_7 the "seventh term" of the sequence, and so on.

We will sometimes start listing sequences at higher indices, like this:

$$a_7, a_8, a_9, a_{10}, \ldots$$

All that really matters is the order of the list, so we can start our indices from any number that's most convenient for us.

Sequences

Formally, a sequence should be thought of as a function.

Definition

A <u>sequence of real numbers</u> is a function $a : \mathbb{N} \to \mathbb{R}$.

Rather than writing $a(n)$ for the n^{th} term of the sequence, we will almost always write a_n, as we did on the previous slide.

For example, consider the function $a : \mathbb{N} \to \mathbb{R}$ given by $a(n) = \frac{1}{n}$.

Some of its values are:

$$a(1) = a_1 = 1, \quad a(2) = a_2 = \frac{1}{2}, \quad \ldots, \quad a(7) = a_7 = \frac{1}{7}, \quad \ldots$$

and so on.

Sequence notation

There are a lot of different notations people commonly use to write down sequences in compact forms. All of these are common ways of denoting the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

- $\{a_n\}_{n=1}^{\infty}$,
- $\{a_n\}$ for short,
- $\{a_n\}_{n\in\mathbb{N}}$
- $(a_n)_{n=1}^{\infty}$,
- (a_n) for short.

I will stick to using the first two.

Some examples of sequences

Some sequences look a lot like the functions we've been studying thusfar:

- $\left\{ \dfrac{1}{2^n} \right\}_{n=1}^{\infty}$.
- $\left\{ n^2 + 7n + 1 \right\}_{n=1}^{\infty}$.
- $\left\{ \dfrac{e^n}{\sin(n)} \right\}_{n=1}^{\infty}$.
- $\left\{ n^n \right\}_{n=1}^{\infty}$.

These all look like the sorts of functions we're familiar with, but with n's in place of x's. We'll return to this idea a bit later.

Some sequences don't look like that, such as $\{n!\}_{n=1}^{\infty}$.

Recursively defined sequences

Something entirely new that can happen with sequences is that they can be *recursively defined*. This is a sequence defined by

- Specifying the value(s) of the first term(s).
- Describing a rule that defines any later term in terms of the previous terms.

Certainly the most famous recursively defined sequence is the *Fibonacci sequence*:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

For this sequence we specify that $f_1 = f_2 = 1$, and then say that for any $n > 2$,

$$f_n = f_{n-1} + f_{n-2}.$$

More ecursive sequences

Example: Consider the sequence defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \log(a_n + 1).$$

Example: Consider the sequence defined by

$$b_1 = 1 \quad \text{and} \quad b_{n+1} = \sqrt[3]{b_n + 6}.$$

Example: Consider the sequence defined by

$$c_1 = 1 \quad \text{and} \quad c_{n+1} = 1 + \frac{1}{c_n + 1}.$$

Exercise: Write out the first few terms of each of these sequences.

Increasing and decreasing sequences

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ is called

- ...increasing if $a_{n+1} > a_n$ for all n (ie. the terms always get bigger).
- ...decreasing if $a_{n+1} < a_n$ for all n (ie. the terms always get smaller).
- ...non-increasing if $a_{n+1} \leq a_n$ for all n (ie. the terms never get bigger).
- ...non-decreasing if $a_{n+1} \geq a_n$ for all n (ie. the terms never get smaller).

A sequence satisfying any of the above four properties is called monotonic or monotone.

A better method, continued.

Determine whether the following sequences are monotonic.

- $7, 7, 7, 7, 7, 7, 7, \dots$
- $\left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$
- $\{\sin(n)\}_{n=1}^{\infty}$.
- $\{sin(n) + n\}_{n=1}^{\infty}$.
- $\left\{ \frac{n^2}{2^n} \right\}_{n=1}^{\infty}$.

You should have noticed that the last sequence is no monotonic, but feels like it should be somehow. We say that sequences like this are eventually monotonic, or that a "tail" of the sequence is monotonic.

Bounded sequences

Another definition that works exactly the same as for functions:

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ is called

- ...bounded above if $\exists M \in \mathbb{R}$ such that $a_n \leq M$ for all n.
- ...bounded below if $\exists m \in \mathbb{R}$ such that $a_n \geq m$ for all n.
- ...bounded if it's bounded above and below.
- ...unbounded if it's not bounded.

Check your understanding: Convince yourself that every non-decreasing sequence is bounded below.

Examples

Example: Show that the sequence given by $a_n = n^{1/n}$ is bounded, and decreasing for $n \geq 3$.

Idea: Use the function $f(x) = x^{1/x}$.

Example: Consider the sequence defined recursively by

$$b_1 = 1 \quad \text{and} \quad b_{n+1} = \sqrt[3]{b_n + 6}.$$

Show that this sequence is bounded and monotonic.

Idea: Prove both of them by induction.

Example: The sequence $a_n = (-1)^n$ is bounded but not monotonic.

Monotone sequences

Question: Suppose f is an increasing function, and we define a sequence by $a_n = f(n)$.

Is $\{a_n\}_{n=1}^{\infty}$ necessarily an increasing sequence?

Question: Suppose f is a function and the sequence $a_n = f(n)$ is increasing.

Is f necessarily an increasing function?

A nice example, that we'll see again later

Consider the sequence $\{H_n\}_{n=1}^{\infty}$ of "harmonic numbers" defined recursively as follows:

$$H_1 = 1 \quad \text{and} \quad H_{n+1} = H_n + \frac{1}{n+1}.$$

- Write down the first few terms of this sequence.
- Convince yourself that this sequence is increasing.
- Convince yourself that for each n,

$$H_n > \int_1^{n+1} \frac{1}{x}\,dx.$$

- From the previous part, it follows that $H_n > \log(n+1)$.
- From the previous part, convince yourself that $\{H_n\}$ is an unbounded sequence.

Convergence

Continuing the analogy between sequences and the functions we've studied before, we now define the limit of a sequence. The definition is *exactly* analogous to the definition of a limit of a function at infinity.

Recall we had the following definition before:

Definition

We say that $\lim_{x \to \infty} f(x) = L$ when

$$\forall \epsilon > 0 \; \exists M \in \mathbb{R} \text{ such that } x > M \implies |f(x) - L| < \epsilon.$$

Convergence

Here's our new definition:

Definition (Sequence convergence)

Let $L \in \mathbb{R}$. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to L if:

$$\forall \epsilon > 0 \; \exists M \in \mathbb{R} \text{ such that } n > M \implies |a_n - L| < \epsilon.$$

If a sequence converges to some L, we say it is convergent. If no such limit exists, we say it is divergent.

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In this case, we write $\lim\limits_{n \to \infty} a_n = L$ or sometimes simply $a_n \to L$.

If a sequence converges to some L, we say it is convergent.

If no such limit exists, we say it is divergent.

Convergence

Example: Let's show that $\lim_{n \to \infty} \frac{1}{2^n} = 0$.

As you might suspect, these limits are closely connected to limits of functions at infinity, the following way:

Theorem

Let f be a function, L a real number, and suppose $\lim_{x \to \infty} f(x) = L$.

Define a sequence by $a_n = f(n)$. Then $\lim_{n \to \infty} a_n = L$ as well.

Proof.

Left as an easy exercise. □

Question: Is the converse true? Think about it.

Limit laws

We won't list them all here, but the familiar limit laws you're aware of for functions also work here.

For example, if $a_n \to L$ and $b_n \to M$, then $a_n + b_n \to L + M$.

This is the analogue of the limit law for sums. The respective analogues are true for constant multiples, products, and quotients.

The Squeeze Theorem also works in this context, and its proof is essentially the same.

An important theorem

This theorem is also an analogue of something we learned long ago about functions: you can "pass a limit through a continuous function".

Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to L.

Let f be a function defined at least at every a_n, and continuous at L.

Then the sequence $\{f(a_n)\}_{n=1}^{\infty}$ converges to $f(L)$.

The catchy way a mathematician would say this is: "Continuous functions respect sequence convergence."

Two main theorems

We are now ready to tackle the two main results of today.

These results describe how boundedness, monotonicity, and convergence relate to one another.

The first one is pretty easy to prove, and you basically proved the second one on your last problem set without realizing it.

Theorem

Every convergent sequence is bounded.

The contrapositive of this theorem is useful enough to be worth stating:

Remark

Every unbounded sequence diverges.

Monotone sequence theorem

Theorem (Monotone Sequence Theorem)

If $\{a_n\}_{n=1}^{\infty}$ *is a bounded, monotonic sequence, then it converges.*

More specifically, if the sequence is non-decreasing, then it converges to $\sup\{\, a_n \,:\, n \in \mathbb{N} \,\}$.

Similarly, if the sequence is non-increasing, then it converges to $\inf\{\, a_n \,:\, n \in \mathbb{N} \,\}$.

Using the MST

Note that the MST doesn't really help you compute limits. It just tells you limits exist sometimes.

It *does* tell you what value the limit should equal (a supremum or infimum), but computing that value usually amounts to doing the same work as proving the limit would involve.

The MST is a very important theorem, and its value is largely theoretical. On the next slide we'll see an example of its use.

Here's something neat

Earlier we discussed the harmonic numbers H_n.

Define a new sequence by $a_n = H_n - \log(n)$.

Fact 1: a_n is bounded below. This is easy to see, as we already know $H_n > \log(n+1)$, and $\log(n+1) > \log(n)$.

Fact 2: $\{a_n\}$ is decreasing:

$$a_n - a_{n-1} = \frac{1}{n} + \log(n) - \log(n-1) = \frac{1}{n} + \log\left(1 - \frac{1}{n}\right) < 0$$

Here's something neat

The sequence $a_n = H_n - \log(n)$ therefore converges by the MST.

The number that it converges to is usually denoted by γ (the Greek letter gamma), and is called *the Euler-Mascheroni constant*.

γ comes up surprisingly often all over mathematics, but surprisingly little is known about it. We don't even know if it's rational or irrational.

We do know that if it is rational, its denominator must be greater than 10^{242080}.

There's at least one whole book written about this number.

Check out the Wikipedia article on this number.