- **Test 3 is tomorrow**, February 3, at 4pm. See the course website for details.
- Today we will:
    - Talk more about integration by parts.
    - Talk about integrating certain combinations of trig functions.
    - Talk a little bit about partial fractions decompositions.
    - Talk about trigonometric substitutions.

Integration by parts

Here's the integration by parts formula, which we derived by integrating the product rule for derivatives at the end of last class.

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Usually we use notation similar to what we used with the substitution rule. We let $u = f(x)$ and $v = g(x)$.

Then accordingly we write $du = f'(x)\,dx$ and $dv = g'(x)\,dx$.

With this notation, the formula looks like this:

$$\int u\,dv = uv - \int v\,du.$$

Integration by parts

$$\int u\,dv = uv - \int v\,du.$$

Note that while the substitution rule actually computed antiderivatives for us, this rule does not.

It simply turns our antiderivative into

> $\langle$*something*$\rangle$ *minus* $\langle$*another antiderivative*$\rangle$.

The "art" of using this formula is choosing u and v in such a way that the new antiderivative on the left side is easier to compute.

Integration by parts examples

Last class we computed $\int xe^x\,dx$ by parts.

First we chose $u = x$ and $dv = e^x\,dx$. This yielded the following:

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$

and we could finish from here easily.

We also tried the other reasonable option: $u = e^x$ and $dv = x\,dx$. This yielded:

$$\int xe^x\,dx = \frac{x^2e^x}{2} - \frac{1}{2}\int x^2e^x\,dx.$$

This, while true, didn't make our lives any easier.

Integration by parts examples

The example on the previous slide is the most "classic" integration by parts example. This next one is the second-most classic example.

Example: Compute $\displaystyle\int \log(x)\,dx$.

Example: Compute $\displaystyle\int \arctan(x)\,dx$ using a similar idea as above.

Example: Compute $\displaystyle\int \sin(\log(x))\,dx$.

Example: Compute $\displaystyle\int e^x \sin(x)\,dx$.

Integrals of certain combinations of trig functions

In this section we're going to talk about some general methods for dealing with certain combinations of trig functions.

There are no new tools to learn here. We'll be using substitution and integration by parts, along with some cleverness with trig identities.

- The Pythagorean identities:
  - $\sin^2(x) + \cos^2(x) = 1$.
  - $\tan^2(x) + 1 = \sec^2(x)$.
- The angle addition identities:
  - $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.
  - $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.
- The double angle formulas (which are easy consequences of the previous two):
  - $\sin(2x) = 2\sin(x)\cos(x)$.
  - $\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$.

Trigonometric integrals

We'll start slowly.

Example: Compute $\displaystyle\int \cos^7(x)\,dx$.

Solution: Use the Pythagorean identity to express all but one power of $\cos(x)$ in terms of $\sin(x)$:

$$\int \cos^7(x)\,dx = \int \left(\cos^2(x)\right)^3 \cos(x)\,dx = \int \left(1-\sin^2(x)\right)^3 \cos(x)\,dx.$$

Now this is easy to do with a substitution: Let $u = \sin(x)$. Then $du = \cos(x)\,dx$, and so we have:

$$\int \cos^7(x)\,dx = \int \left(1-\sin^2(x)\right)^3 \cos(x)\,dx. = \int (1-u^2)^3\,du,$$

which is now a polynomial, which we can integrate easily.

Generalize the previous result!

Notice that nothing was special about the number 7 in the previous result.

All that actually mattered was that it was odd, so we could split off one power of $\cos(x)$, and express the remaining ones as $\cos^2(x)$ to some power.

So in general, for any odd number $2k+1$, where k is a natural number, we have:

$$\int \cos^{2k+1}(x)\,dx = \int \left(1-u^2\right)^k\,du.$$

The only difference if we start with an odd power of $\sin(x)$ is that when $u=\cos(x)$, we'll have $du = -\sin(x)\,dx$. There's an extra minus sign. In that case, we get:

$$\int \sin^{2k+1}(x)\,dx = -\int \left(1-u^2\right)^k\,du.$$

Slightly more complicated

What if there are sines and cosines!?

Example: Compute $\int \sin^5(x)\cos^{17}(x)\,dx$.

Solution: Same idea as before. Use the Pythagorean identity to express the integral as

$$\int \Big(\text{stuff in terms of } \sin(x)\Big)\cos(x)\,dx$$

or

$$\int \Big(\text{stuff in terms of } \cos(x)\Big)\sin(x)\,dx.$$

In this case, we can do either one.

Slightly more complicated, continued.

This is our integral again: $\displaystyle\int \sin^5(x)\cos^{17}(x)\,dx$.

We can split off all but one power of $\cos(x)$ to get:

$$\int \sin^5(x)\left(\cos^2(x)\right)^8\cos(x)\,dx = \int \sin^5(x)\left(1-\sin^2(x)\right)^8\cos(x)\,dx.$$

Then the substitution $u=\sin(x)$ will get us to:

$$\int u^5\left(1-u^2\right)^8\,du$$

which is a polynomial.

Slightly more complicated, continued.

This is our integral again: $\int \sin^5(x) \cos^{17}(x)\, dx$.

We can also split of all but one power of $\sin(x)$ to get:

$$\int \sin(x) \left(\sin^2(x)\right)^2 \cos^{17}(x)\, dx = \int \sin(x) \left(1 - \cos^2(x)\right)^2 \cos^{17}(x)\, dx.$$

Then the substitution $u = \cos(x)$ will get us to:

$$\int (1 - u^2)^2 u^{17}\, du$$

which is again a polynomial.

Generalize!

This is our integral again: $\int \sin^5(x)\cos^{17}(x)\,dx$

What did we actually need to be able to do what we did above?

The 5 and 17 weren't specifically important. We just needed them to be odd, like before.

Moreover, we only needed one of them to be odd.

So now we can compute (in principle) any integral of the form:

$$\int \cos^m(x)\sin^n(x)\,dx$$

as long as at least one of n or m is odd.

The same idea but with tangent and secant.

Our main tool above was that we could use the identity

$$\sin^2(x) + \cos^2(x) = 1$$

to express even powers of $\sin(x)$ in terms of even powers of $\cos(x)$, and vice versa.

We also know a similar identity for tangent and secant:

$$\tan^2(x) + 1 = \sec^2(x).$$

This identity lets us do the same sort of thing.

The difference here is in the derivatives.

$$\frac{d}{dx}\tan(x) = \sec^2(x) \quad \text{and} \quad \frac{d}{dx}\sec(x) = \sec(x)\tan(x).$$

Examples with tangent and secant.

Try the following examples:

Example: Compute $\int \sec^{12}(x)\,dx$.

Example: Compute $\int \tan^7(x)\sec^7(x)\,dx$.

Can you generalize these results?

Similar methods allow us to integrate

- Any even power of $\sec(x)$,
- Any product of the form $\tan^m(x)\sec^n(x)$ in which...
    - ...at least one of m or n is odd, or
    - ...n is even.

What's left?

Even powers seem to give us trouble. What about even powers of just $\sin(x)$ or $\cos(x)$?

Double angle formulas take care of that for us.

Example: Compute: $\displaystyle\int \sin^2(x)\, dx$.

Solution: Use the double angle identity for cosine that involves $\sin^2(x)$:

$$\int \sin^2(x)\, dx = \int \frac{1}{2}\left(1 - \cos(2x)\right)\, dx = \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right] + C$$

Of course you can do the same thing with $\cos^2(x)$ using the identity

$$\cos^2(x) = \frac{1}{2}\left(1 + \sin(2x)\right).$$

What about *products* of even powers?

This seems bad, but all is not lost.

Example: Compute $\int \sin^2(x)\cos^2(x)\,dx$.

Solution: Use the double angle formula for $\sin(x)$ first!

$$\int (\sin(x)\cos(x))^2\ dx = \int \left(\frac{1}{2}\sin(2x)\right)^2\ dx = \frac{1}{4}\int \sin^2(2x)\,dx$$

Then the substitution $u = 2x$ will get you to

$$\frac{1}{8}\int \sin^2(u)\,du,$$

which you now know how to do.

We'll stop here...

So as you can see, there are no new tools here. Just substitution, a few simple trig identities, and some cleverness.

What's left to talk about? Here's one thing which will lead us to the next topic:

Example: Compute $\int \sec(x)\,dx$.

Method 1: This is a dirty trick that I would never expect you to notice yourself.

$$\int \sec(x)\,dx = \int \sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right)\,dx$$

Now let $u = \sec(x) + \tan(x)$, and the numerator of the fraction is precisely $u'(x)$.

Integral of secant

$$\int \sec(x)\,dx = \int \sec(x)\left(\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\right)\,dx$$

Now let $u = \sec(x) + \tan(x)$, and the numerator of the fraction is precisely $u'(x)$. So we get

$$\int \sec(x)\,dx = \int \frac{du}{u} = \log|u| + C.$$

And so

$$\int \sec(x)\,dx = \log|\sec(x)+\tan(x)| + C.$$

A better method.

Here's another way of approaching the same antiderivative that's not a dirty trick, and which will be more useful for us.

$$\int \sec(x)\,dx = \int \frac{1}{\cos(x)}\,dx = \int \frac{\cos(x)}{\cos^2(x)}\,dx = \int \frac{\cos(x)}{1-\sin^2(x)}\,dx$$

This should remind you of some of the stuff we did earlier.

Let $u = \sin(x)$. Then we have:

$$\int \sec(x)\,dx = \int \frac{du}{1-u^2}.$$

We'll concentrate on this new integral for a minute.

A better method, continued.

We just showed that if $u = \sin(x)$, then

$$\int \sec(x)\,dx = \int \frac{du}{1-u^2} = \int \frac{du}{(1-u)(1+u)}.$$

Exercise: Find real numbers A and B such that

$$\frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}.$$

Solution: $A = B = \frac{1}{2}$. With this, we can solve the integral.

A better method, continued

So, now we have that if $u = \sin(x)$,

$$\int \sec(x)\,dx = \frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u}\,du.$$

Each of these terms can be integrated easily, yielding:

$$\int \sec(x)\,dx = \frac{1}{2}\Big[\log|1+u| - \log|1+u|\Big] + C$$

We can simplify this, and put it back in terms of x:

$$\int \sec(x)\,dx = \frac{1}{2}\log\left|\frac{1+\sin(x)}{1-\sin(x)}\right| + C.$$