MAT137 - Term 2, Week 4

- Reminders:
  - Your Problem Set 6 is due **tomorrow** at 3pm.
  - **Test 3 is next Friday**, February 3, at 4pm. See the course website for details.
- Today we will:
  - Talk more about substitution.
  - Talk about computing volumes of solids of revolution.
  - Introduce integration by parts.

Substitution

Recall that the technique of substitution is derived from integrating the chain rule.

Here's the chain rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x).$$

Integrating this yields:

$$f(g(x)) + C = \int f'(g(x))\, g'(x)\, dx.$$

Substitution

$$f(g(x)) + C = \int f'(g(x))\, g'(x)\, dx.$$

To use this formula to compute the antiderivative $\int h(x)\, dx$, you must find two functions f' and g such that

$$h(x) = f'(g(x))\, g'(x).$$

Once you have these functions, all you need to figure out is f (which is an antiderivative of f').

Sometimes this is easy, like in the case of

$$\int 2x(x^2 + 1)^7\, dx.$$

In this case $g(x)$ should be $x^2 + 1$ and $f'(x)$ should be x^7, and so $f(x) = \frac{x^8}{8}$.

Substitution

Sometimes (most times, sadly) it isn't quite so easy, and we have to adjust some things.

Example: Compute $\int \cos^2(7x)\sin(7x)\,dx$.

Substitution notation

When using the substitution rule, we will use the notation

$$u = g(x),$$

and

$$du = g'(x)\,dx.$$

With this notation, the substitution rule says:

$$f(u) + C = \int f'(u)\,du,$$

which is something we already know from the FTC.

This process amounts to changing the variable from x to something that's more convenient for us to integrate with.

Substitution

Some strategy for using substitution.

1. Look at your integrand, and try to find an occurence of a function $g(x)$ and its derivative $g'(x)$. You may need to try several things before one works. Remember that if you're just missing a multiplicative constant, you can adjust for it manually.
2. Let $u = g(x)$ be your new variable, and then compute $du = g'(x)\,dx$.
3. Express the whole integrand in terms of u and du.
4. Compute the antiderivative (which will be doable, if you chose u well).
5. Put everything back in terms of x at the end.

Substitution examples

Example: Compute $\displaystyle\int e^{7x+5}\,dx$.

Example: Compute $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$.

Example (trickier): Compute $\displaystyle\int x\sqrt[3]{x+3}\,dx$.

You **must** do lots of examples to develop intuition for this. Make sure you do all the practise problems for this section (5.7).

Definite integrals and substitution

We must be a bit careful when using substitution to compute definite integrals.

There are two ways to do this.

The first is foolproof, but might take more work.

The second requires remembering something, but is usually much easier (computationally speaking).

Definite integrals and substitution: method 1

First, use substitution to find the indefinite integral, as we have been doing.

Then apply FTC2 to compute the definite integral as you would usually do.

Example: Compute $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)\,dx$.

Definite integrals and substitution: method 1

Change the limits of integration when you change the variable.

Theorem

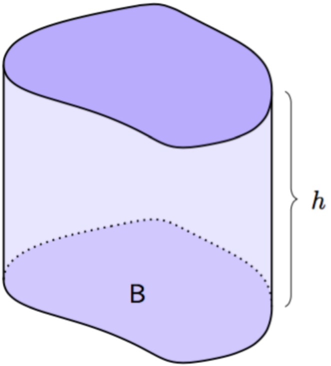
If $u = g(x)$, then

$$\int_a^b f'(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Example: Compute the same intedefinite integral from the previous slide using this method.

Recall how you find the volume of a cylinder or triangular prism. In both cases you find the area of the "base" shape (a circle or triangle, respectively), and then multiply by the height.

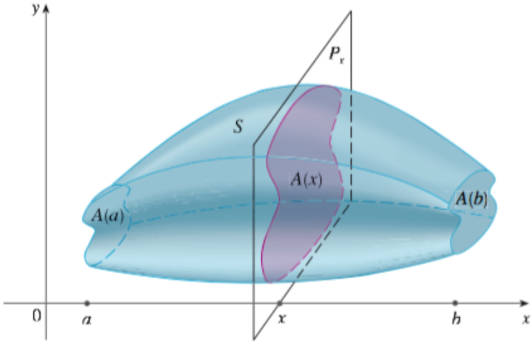
In the case of a more general solid like this, the idea is the same:



$$A = \text{area}(B) \cdot h$$

Volumes

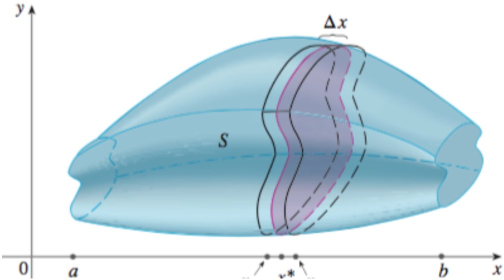
In the case of more complicated solids like this:



The situation is not as simple, but the methods of integration allow us to find the answer.

Volumes

Imagine partitioning up the interval $[a, b]$ into n pieces. In the i^{th} subinterval, pick a point x_i^*.



Approximate the volume of the shape over the i^{th} subinterval by assuming the cross-sectional area on it is always $A(x_i^*)$.

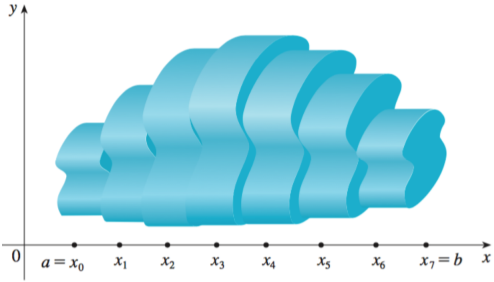
Then the volume of this approximation over the i^{th} subinterval is $A(x_i^*)\Delta x_i$.

Volumes

If we do this for every subinterval, we get the following approximation of the volume V:

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i.$$

This corresponds to the volume of a shape like this:



Of course, this should remind you of a Riemann sum.

Volumes

The finer these partitions get, the closer this approximation should get to the actual volume V.

Accordingly, this means we should get:

$$V = \int_a^b A(x)\,dx$$

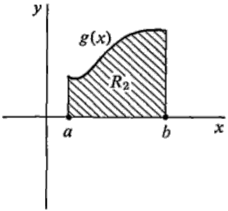
(where again $A(x)$ is the cross-sectional area of the solid at the point x).

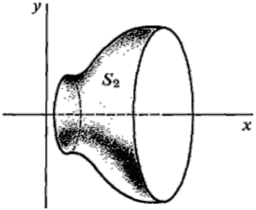
Example: Consider the solid whose base is the region bounded between $y = x$ and $y = x^2$, and whose cross-sections parallel to the y-axis are squares. Compute its volume.

Volume examples

The easiest examples of solids whose volumes you can compute in this way are "solids of revolution".

These are obtained by taking a region on the plane (usually bounded between two curves), and rotating it about some axis to obtain a solid.

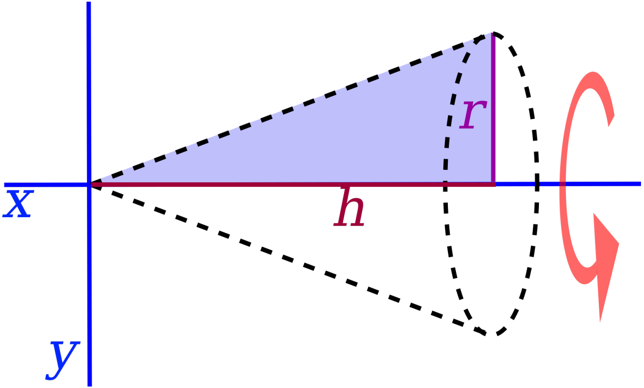




Volumes

Example: Find the volume of the right circular cone with base radius r and height h.

To use this method, we first realize the cone as a solid of revolution:



Volume examples

Example: Consider again the region bounded between $y = x$ and $y = x^2$. Rotate this region around the x-axis to form a solid. Compute its volume.

Example: Take the same region as above, but now rotate it around the y-axis. Compute the volume of the resulting solid.

Example: Use the same region again, but now rotate it about the line $x = -1$. Compute the volume of the resulting solid.

Another way of slicing things up

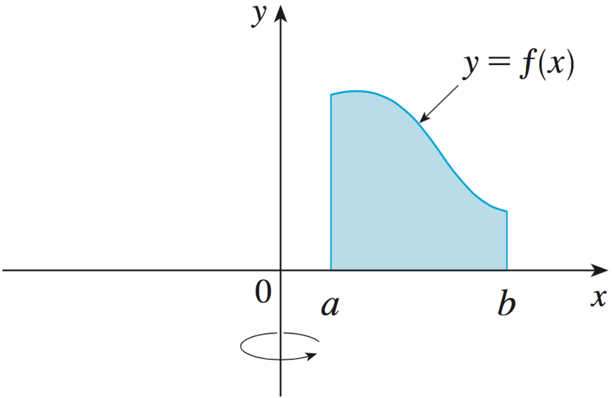
So far we've approximated the volumes of shapes by dividing them into a series of discs.

Another way to think of this is that we took our solid, and made a bunch of straight cuts in it, then approximated the volume of each piece.

Instead of straight cuts, we can make circular cuts. Think of an apple corer:

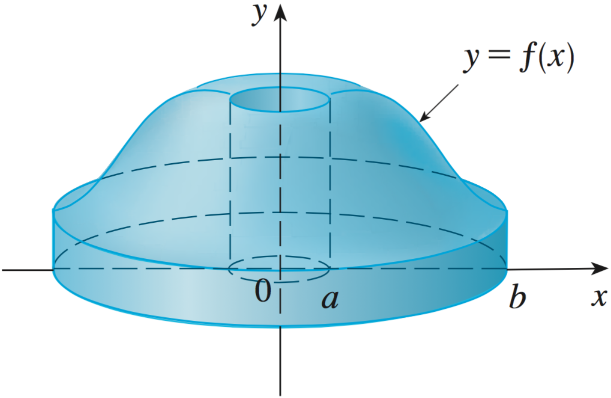
Another way of slicing

Suppose we start with a region like this:



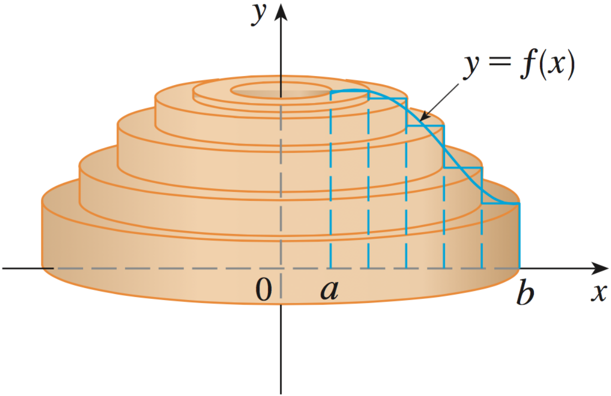
Another way of slicing

And rotate it about the y-axis to obtain a solid like this:



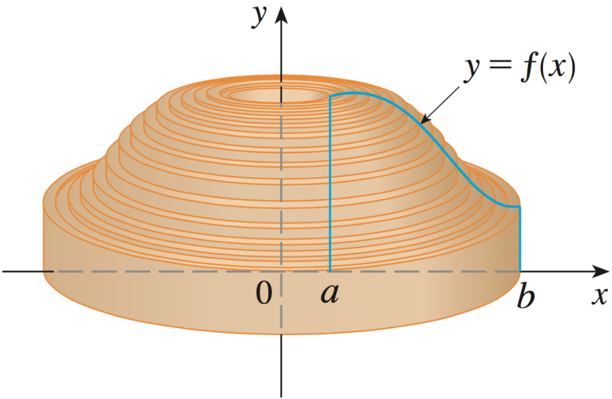
Another way of slicing

We can divide $[a, b]$ into five subintervals and get an approximation like this:



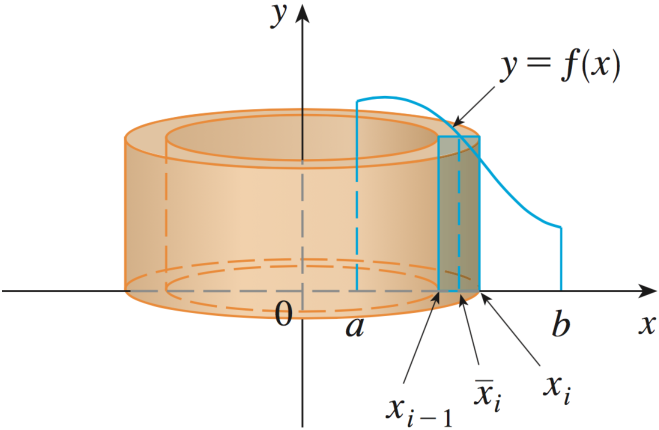
Another way of slicing

If we divide $[a, b]$ into many more subintervals, we get an approximation like this:



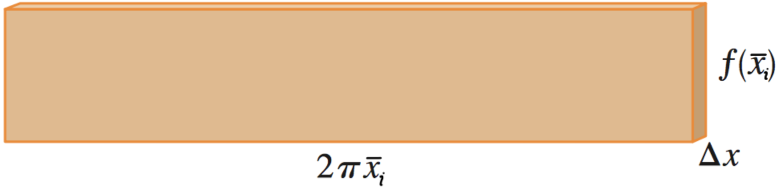
Another way of slicing

Going back to the coarser approximation, we can look at just one of the "slices":



Another way of slicing

Since we're imagining this slice being *very* thin, we can "unroll" it:



The volume of this shape is: $2\pi\overline{x_i} \cdot f(\overline{x_i}) \cdot \Delta x_i$.

Another way of slicing

Again, this should remind you of a Riemann sum.

If we do this for every slice, we obtain an approximation for the volume V:

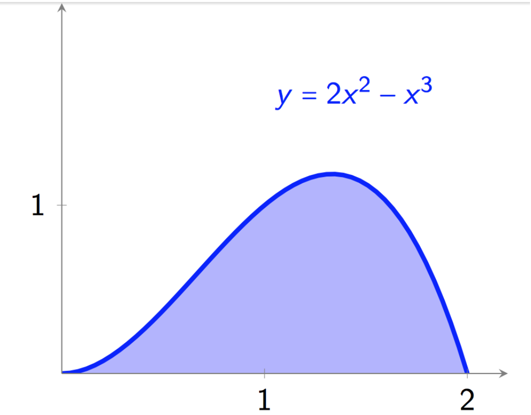
$$V \approx \sum_{i=1}^{n} 2\pi \overline{x_i} \cdot f(\overline{x_i}) \cdot \Delta x_i$$

This <u>is</u> a Riemann sum. This suggests the actual volume should equal:

$$V = \int_a^b 2\pi x\, f(x)\, dx.$$

Example

Consider the region between the curve $y = x^2 - x^3$ and the x-axis. Rotate this region around the y-axis, and compute the volume of the resulting shape.



Example

Earlier you compute the volume of the solid obtained by rotating the region bounded between $y = x$ and $y = x^2$ about the y-axis.

Compute this volume again, using this new method of slicing up the shape.

Homework: Use either (or both!) of these methods to derive the formula for the volume of a sphere.

Integration by parts

Earlier we saw that the substitution method was obtained by integrating the chain rule.

Another powerful tool for computing antiderivatives can be obtained from integrating the product rule.

Here's how the product rule usually looks:

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

We can rearrange it like this:

$$f(x)g'(x) = \frac{d}{dx}f(x)g(x) - f'(x)g(x).$$

Integration by parts

$$f(x)g'(x) = \frac{d}{dx}f(x)g(x) - f'(x)g(x).$$

We can then integrate both sides of this equation to the following, which is called the Integration by Parts formula:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Usually we use notation similar to what we used with the substitution rule. We let $u = f(x)$ and $v = g(x)$.

Then accordingly we write $du = f'(x)\,dx$ and $dv = g'(x)\,dx$.

With this notation, the formula looks like this:

$$\int u\,dv = uv - \int v\,du.$$

Integration by parts

$$\int u\,dv = uv - \int v\,du.$$

Note that while the substitution rule actually computed antiderivatives for us, this rule does not.

It simply turns our antiderivative into

> $\langle$*something*$\rangle$ *minus* $\langle$*another antiderivative*$\rangle$.

The "art" of using this formula is choosing u and v in such a way that the new antiderivative on the left side is easier to compute.