- Today we will:
  - Prove both parts of the FTC and do some related exercises.
  - Talk about areas a bit.
  - Introduce integration by substitution.

The FTC

Theorem (Fundamental Theorem of Calculus)

1. *Suppose f is integrable on $[a, b]$, and $c \in [a, b]$. Then*

$$F_c(x) = \int_c^x f(t)\,dt$$

is continuous on $[a, b]$.

F_c is differentiable at any point x where f is continuous, and at such a point $F_c'(x) = f(x)$.

2. *Let f be continuous on $[a, b]$. If F is any antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Two more properties of definite integrals

Before we can prove the FTC, we need two more basic properties of definite integrals.

Before those, let's first remind ourselves of the following very important property, which we will also use:

Proposition (Additivity of Domain)

For any real numbers a, b, c for which the following expression makes sense for f, it is true:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Monotonicity and Subnormality

Here are our two new properties:

Proposition (Monotonicity)

If f and g are integrable on $[a, b]$, and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

The proof of this is left as an exercise. It should be pretty intuitive.

Proposition (Subnormality)

If f is integrable on $[a, b]$, then $|f|$ is also integrable on $[a, b]$, and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

This property should remind you of the Triangle Inequality.

Proof of subnormality

We'll just prove the inequality, and skip showing that $|f|$ is integrable.

Note that by definition of the absolute value, we know the following for all $x \in [a, b]$:

$$-|f(x)| \leq f(x) \leq |f(x)|.$$

Integrate all three terms, and use monotonicty to get:

$$-\int_a^b |f(x)|\ dx. \leq \int_a^b f(x)\, dx. \leq \int_a^b |f(x)|\ dx.$$

From this it follows immediately that

$$\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\ dx.$$

Remind yourself about F_c

Before we prove FTC1, let's take a moment to remind ourselves about the function F_c that it mentions.

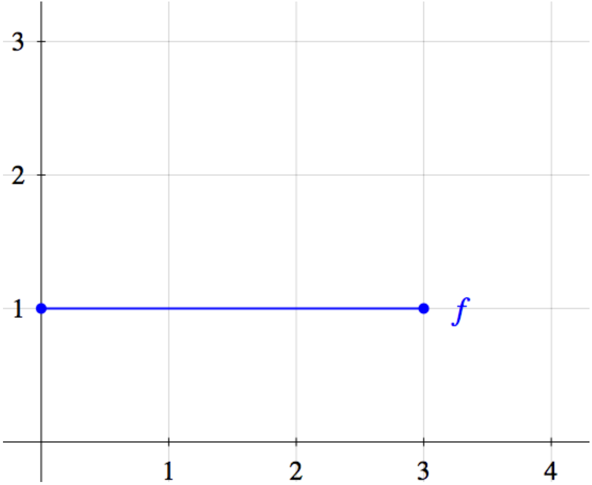
Suppose f is integrable on $[a, b]$, and $c \in [a, b]$. Then we define

$$F_c(x) = \int_c^x f(t)\,dt.$$

- The value of $F_c(x)$ corresponds to the area under the graph of f between the fixed number c, and x. This is a function of x.
- Recall that for any c and d in $[a, b]$, F_c and F_d differ by only a constant. The value of that constant is $\int_c^d f(x)\,dx$.

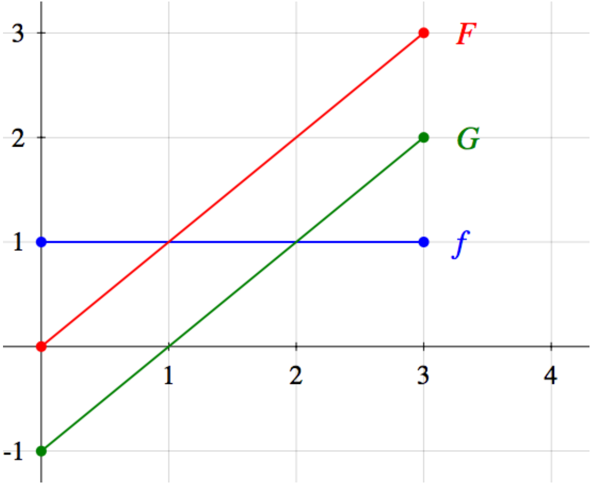
Remind yourself about F_c

Exercise: Sketch the graphs of $F(x) = \int_0^x f(t)\,dt$ and $G(x) = \int_1^x f(t)\,dt$



Answer to previous exercise

Exercise: Sketch the graphs of $F(x) = \int_0^x f(t)\,dt$ and $G(x) = \int_1^x f(t)\,dt$



Proof of FTC1

Let's restate FTC1 for convenience.

Theorem

Suppose f is integrable on $[a, b]$, and let $c \in [a, b]$. Then

$$F_c(x) = \int_c^x f(t)\,dt$$

is continuous on $[a, b]$.

F_c is differentiable at any point x where f is continuous, and at such a point $F'(x) = f(x)$.

Proof of FTC1

We're actually going to prove a special case: We will assume f is continuous on $[a, b]$, and prove F_c is differentiable on (a, b) and $F_c'(x) = f(x)$.

This will suffice for our purposes in this course. The general proof is not much more complicated, and can be found in Tyler's notes.

So, suppose f is continuous on $[a, b]$, and fix a point $x \in (a, b)$.

We want to show that $F_c'(x) = f(x)$.

The definition of this statement is:

$$\lim_{h \to 0} \frac{F_c(x+h) - F_c(x)}{h} - f(x) = 0.$$

Proof of FTC1

$$\lim_{h\to 0} \frac{F_c(x+h)-F_c(x)}{h} - f(x) = 0.$$

We'll prove the this using the definition of the limit:

For all $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 < |h| < \delta \Longrightarrow \left| \frac{F_c(x+h)-F_c(x)}{h} - f(x) \right| < \epsilon$$

We'll begin to tackle this by manipulating the part inside the absolute values.

Proof of FTC1

First note that using the definition of F_c we have:

$$F_c(x+h) - F_c(x) = \int_c^{x+h} f(t)\,dt - \int_c^{x} f(t)\,dt.$$

Using additivity of domain this reduces to:

$$F_c(x+h) - F_c(x) = \int_x^{x+h} f(t)\,dt.$$

Proof of FTC1

Next we'll note something slightly tricky:

$$\int_x^{x+h} f(x)\,dt = h\,f(x).$$

This is because $f(x)$ is a fixed constant inside that integral, so the definite integral represents the area of a rectangle with height $f(x)$ and width h.

The equation above rearranges to this:

$$f(x) = \frac{1}{h}\int_x^{x+h} f(x)\,dt.$$

Proof of FTC1

Combining these two results, we now have:

$$\left|\frac{F_c(x+h)-F_c(x)}{h}-f(x)\right| = \frac{1}{|h|}\left|\int_x^{x+h} f(t)\,dt - \int_x^{x+h} f(x)\,dt\right|.$$

Using linearity of definite integrals, this reduces to:

$$\left|\frac{F_c(x+h)-F_c(x)}{h}-f(x)\right| = \frac{1}{|h|}\left|\int_x^{x+h} f(t)-f(x)\,dt\right|.$$

And using subnormality (the Triangle Inequality-like property) this gives us:

$$\left|\frac{F_c(x+h)-F_c(x)}{h}-f(x)\right| \leq \frac{1}{|h|}\int_x^{x+h} |f(t)-f(x)|\,dt.$$

Proof of FTC1

$$\left| \frac{F_c(x+h) - F_c(x)}{h} - f(x) \right| \leq \frac{1}{|h|} \int_x^{x+h} |f(t) - f(x)| \, dt.$$

So it will suffice for us to show the following:

For all $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 < |h| < \delta \Longrightarrow \frac{1}{|h|} \int_x^{x+h} |f(t) - f(x)| \, dt < \epsilon$$

To do this, we will use the fact that f is continuous at x to make the $|f(t) - f(x)|$ term on the right small enough to make this true.

Proof of FTC1

We begin as always be fixing an arbitrary $\epsilon > 0$.

Since f is continuous at x, we can find a $\delta > 0$ such that

$$0 < |x - t| < \delta \Longrightarrow |f(t) - f(x)| < \epsilon.$$

We claim that this same δ works for the limit we are trying to prove.

So, we assume $0 < |h| < \delta$, and try to prove that

$$\frac{1}{|h|} \int_{x}^{x+h} |f(t) - f(x)| \, dt < \epsilon.$$

Proof of FTC1

$$\frac{1}{|h|}\int_x^{x+h} |f(t)-f(x)|\, dt < \epsilon.$$

Since $|h| < \delta$, for all $t \in [x, x+h]$ we have that $|f(t) - f(x)| < \epsilon$.

Therefore, by monotonicity of definite integrals, we get:

$$\frac{1}{|h|}\int_x^{x+h} |f(t)-f(x)|\, dt < \frac{1}{|h|}\int_x^{x+h} \epsilon\, dt.$$

Since ϵ is a fixed positive constant, the integral on the right represents the area of a rectangle. We compute:

$$\int_x^{x+h} \epsilon\, dt = |h|\,\epsilon.$$

Proof of FTC1

Combining everything we have, we see that we're finished:

Suppose $0 < |h| < \epsilon$. Then

$$\begin{aligned}
\left| \frac{F_c(x+h) - F_c(x)}{h} - f(x) \right| &\leq \frac{1}{|h|} \int_x^{x+h} |f(t) - f(x)| \, dt \\
&< \frac{1}{|h|} \int_x^{x+h} \epsilon \, dt \\
&= \frac{1}{|h|} |h| \epsilon \\
&= \epsilon
\end{aligned}$$

We did it!

Indefinite integrals

FTC1 tells us that definite integrals can be used to define antiderivatives. That connection justifies the following definition.

Definition

Let f be a continuous function on an interval I. Then we denote by

$$\int f(x)\,dx$$

the collection of *all* antiderivatives of f.

Note the difference in notation:

- $F_c(x) = \int_c^x f(t)\,dt$ is one particular antiderivative of f.
- $\int f(x)\,dx$ is all antiderivatives of f.

Also recall that any two antiderivatives of f differ from one another by a constant.

Using FTC1

The main thing we'll use FTC1 for is proving FTC2. But it does have some applications of its own.

Example: Find the derivative of $F(x) = \int_0^x \cos^2(t)\, dt$.

Example: Find the derivative of $G(x) = \int_x^7 7te^{t^2}\, dt$.

Example (trickier): Find the derivative of $H(x) = \int_0^{x^2} \frac{\sin(t)}{t}\, dt$.

Hint 1: The answer is not $\frac{\sin(x^2)}{x^2}$.

Hint 2: Try to express $H(x)$ in terms of $F_0(x) = \int_0^x \frac{\sin(t)}{t}\, dt$.

Using FTC1

Once you know the general idea of the problems above, they all become straightforward.

Example: Find the derivative of $\int_{x^2}^{\sin(x)} \arctan(t)\, dt$

Step 1: Use additivity of domain to say:

$$\int_{x^2}^{\sin(x)} \arctan(t)\, dt = \int_{x^2}^{0} \arctan(t)\, dt + \int_{0}^{\sin(x)} \arctan(t)\, dt$$

Now it's like a combination of the previous problems. Finish this example as homework.

Proof of FTC2

We'll now prove FTC2, which is quite straightforward now that we know FTC1. We'll restate the theorem here:

Theorem

Let f be continuous on $[a, b]$. If F is <u>*any*</u> *antiderivative of f on $[a, b]$, then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Proof of FTC2

Fix any $c \in [a, b]$. By FTC1, we know that one particular antiderivative of f on $[a, b]$ is:

$$F_c(x) = \int_c^x f(t)\,dt.$$

Now, using additivity of domain, we can see:

$$\begin{aligned}
\int_a^b f(t)\,dt &= \int_a^c f(t)\,dt + \int_c^b f(t)\,dt \\
&= -\int_c^a f(t)\,dt + \int_c^b f(t)\,dt \\
&= -F_c(a) + F_c(b) \\
\\
&= F_c(b) - F_c(a)
\end{aligned}$$

Proof of FTC2

So we have directly proved the result for the particular antiderivative F_c. Now suppose G is any other antiderivative of f on $[a, b]$.

We already know that G and F_c must differ by a constant.

Let's say $F_c(x) = G(x) + D$ for some real number D.

Then we have:

$$\int_a^b f(t)\,dt = F_c(b) - F_c(a) = (G(b) + D) - (G(a) + D) = G(b) - G(a).$$

Some notation

To make our lives a bit easier, we will use the following standard notation:

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

So for example:

$$\Big[7\sin(x^2)\Big]_0^3 = 7\sin(3^2) - 7\sin(0).$$

Using FTC2 to compute definite integrals

Example: Compute $\displaystyle\int_0^{\pi} \sin(x)\,dx$.

Example: Compute $\displaystyle\int_1^{2} 3x^2 + 2x + 1$.

Thanks to FTC2, computing definite integrals reduces entirely to finding antiderivatives. We'll talk a lot about finding antiderivatives later, and develop some theory to help us do that.

But first, we'll talk a bit about areas.

So far we've been saying that definite integrals compute areas under curves. That's not *exactly* true.

Example: Using FTC2, compute $\int_0^{2\pi} \sin(x)\, dx$.

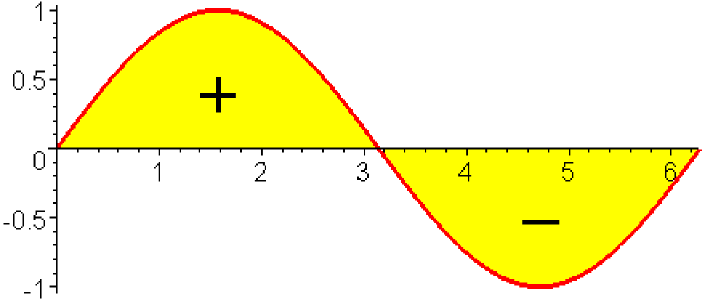
The answer you obtained was 0. But clearly there is area under the graph. So what's going on?

Signed area

The answer is that definite integrals actually compute something called signed area.

That means they assign a negative value to areas *underneath* the x-axis.

So this is what happend with the calculation you did above:



Total area

To compute the *total* (ie. *unsigned*) area under the graph of a function, we "force" it to be above the x-axis, in the following sense:

$$\text{Total area under } f \text{ between } a \text{ and } b = \int_a^b |f(x)|\, dx.$$

Areas between curves example

Example: Compute the total bounded area between the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$.

If we call this area A, then we can see from a picture that:

$$A = \int_0^1 \sqrt{x} - x^2 \, dx$$

Then we compute that $A = \frac{1}{3}$.

A similar example

Compute the total area bounded between the same two curves $f(x) = \sqrt{x}$ and $g(x) = x^2$, between $x = 0$ and $x = 2$.

If we just compute

$$\int_0^2 \sqrt{x} - x^2 \, dx$$

we won't get the correct answer, since $\sqrt{x} - x^2$ is negative between 1 and 2.

The solution is to compute the following:

$$\int_0^2 |\sqrt{x} - x^2| \, dx = \int_0^1 \sqrt{x} - x^2 \, dx + \int_1^2 x^2 - \sqrt{x} \, dx.$$

Complete the exercise from here as homework.

Area between curves example

Compute the area enclosed between the curves $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$.

First, we find the two points of intersection by solving:

$$y + 1 = \frac{1}{2}y^2 - 3.$$

The two points are $(-1, -2)$ and $(5, 4)$.

Method 1: Integrate along the x-axis.

$$\int_{-3}^{-1} \sqrt{2x+6} - (-\sqrt{2x+6})\, dx + \int_{-1}^{5} \sqrt{2x+6} - (x-1).$$

Tricky part: What is an antiderivative of $\sqrt{2x+6}$?

Example, continued

Method 2 (the smarter way): Integrate along the y-axis.

$$\int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right)\, dy$$

Answer: 18